Theoretical Basis of GVAR: Mathematical Foundations and Assumptions

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Part 1:

The GVAR Modeling Framework

A simple VARX* structure

- At the core of the GVAR modelling framework lies a set of individual VARX* models, combined in such a way to give rise to the global VAR model.

- Consider a set of countries $i = 0, 1, 2, \ldots, N$; with country 0 taken as the reference country.

- For country i, abstracting from deterministics and higher order lags, consider the $VARX^*(1,1)$ structure:

$$\mathbf{x}_{it} = \mathbf{\Phi}_i \mathbf{x}_{i,t-1} + \mathbf{\Lambda}_{i0} \mathbf{x}^*_{it} + \mathbf{\Lambda}_{i1} \mathbf{x}^*_{i,t-1} + \mathbf{u}_{it},$$

$$\mathbf{x}_{it} : k_i \times 1 \text{ vector of domestic variables}$$

$$\mathbf{x}^*_{it} : k^*_i \times 1 \text{ vector of foreign variables}$$

- where $\mathbf{x}^*_{it} = \sum_{j=0}^{N} w_{ij} \mathbf{x}_{jt}, w_{ii} = 0,$

A simple VARX* structure

- with $w_{ij}, j = 0, 1, \ldots, N$ a set of weights such that $\sum_{j=0}^{N} w_{ij} = 1$,
- and $\mathbf{u}_{it}$ are cross sectionally weakly correlated such that $\overline{\mathbf{u}}_{it} = \sum_{j=0}^{N} w_{ij}\mathbf{u}_{jt} \xrightarrow{p} \mathbf{0}$, as $N \to \infty$ (where $\xrightarrow{p}$ denotes convergence in probability).
- The domestic and foreign variable vectors could contain, for example, the following variables:

$$\mathbf{x}_{it} = \begin{pmatrix} y_{it} \\ \Delta p_{it} \\ \rho_{it}^{S} \end{pmatrix}, \quad \mathbf{x}_{it}^{*} = \begin{pmatrix} y_{it}^{*} \\ \Delta p_{it}^{*} \\ \rho_{it}^{S*} \end{pmatrix}$$

A simple VARX* structure

- where

$$y_{it} = \ln\left(GDP_{it}/CPI_{it}\right),$$
$$\Delta p_{it} = p_{it} - p_{i,t-1}, p_{it} = \ln\left(CPI_{it}\right),$$
$$\rho_{it}^{S} = 0.25\ln\left(1 + R_{it}^{S}/100\right),$$

- GDP_{it} is the nominal gross domestic product, CPI_{it} the consumer price index, R_{it}^{S} is the annualized short rate, and

$$y_{it}^{*} = \Sigma_{j=0}^{N} w_{ij} y_{jt},$$
$$\Delta p_{it}^{*} = \Sigma_{j=0}^{N} w_{ij} \Delta p_{jt},$$
$$\rho_{it}^{S*} = \Sigma_{j=0}^{N} w_{ij} \rho_{jt}^{S}.$$

Estimation of the individual VARX* models

- For country i, consider the VARX$^*(2,2)$ structure

$$\begin{aligned}\mathbf{x}_{it} =& \mathbf{a}_{i0} + \mathbf{a}_{i1}t + \mathbf{\Phi}_{i1}\mathbf{x}_{i,t-1} + \mathbf{\Phi}_{i2}\mathbf{x}_{i,t-2} + \mathbf{\Lambda}_{i0}\mathbf{x}^*_{it} \\ &+ \mathbf{\Lambda}_{i1}\mathbf{x}^*_{i,t-1} + \mathbf{\Lambda}_{i2}\mathbf{x}^*_{i,t-2} + \mathbf{u}_{it}\end{aligned}$$

- The error correction form (VECMX*) of the VARX$^*(2,2)$ specification can be written as

$$\Delta\mathbf{x}_{it} = \mathbf{c}_{i0} - \alpha_i\boldsymbol{\beta}'_i\left[\mathbf{z}_{i,t-1} - \gamma_i(t-1)\right] + \mathbf{\Lambda}_{i0}\Delta\mathbf{x}^*_{it} + \Gamma_i\Delta\mathbf{z}_{i,t-1} + \mathbf{u}_{it},$$

  where $\mathbf{z}_{it} = \left(\mathbf{x}'_{it}, \mathbf{x}^{*\prime}_{it}\right)'$, $\boldsymbol{\alpha}_i$ is a $k_i \times r_i$ matrix of rank r_i and $\boldsymbol{\beta}_i$ is a $(k_i + k_i^*) \times r_i$ matrix of rank r_i.

- By partitioning $\boldsymbol{\beta}_i$ as $\boldsymbol{\beta}_i = \left(\boldsymbol{\beta}'_{ix}, \boldsymbol{\beta}'_{ix*}\right)'$ conformable to $\mathbf{z}_{it}$, the r_i error correction terms defined by the above equation can be written as: $\boldsymbol{\beta}'_i\left(\mathbf{z}_{it} - \gamma_i t\right) = \boldsymbol{\beta}'_{ix}\mathbf{x}_{it} + \boldsymbol{\beta}'_{ix*}\mathbf{x}^*_{it} - \left(\boldsymbol{\beta}'_i\gamma_i\right)t,$

Estimation of the individual VARX* models

- which allows for the possibility of cointegration both within $\mathbf{x}_{it}$ and between $\mathbf{x}_{it}$ and $\mathbf{x}_{it}^*$, and consequently across $\mathbf{x}_{it}$ and $\mathbf{x}_{jt}$ for $i \neq j$.
- Assumption: For estimation, $\mathbf{x}_{it}^*$ are treated as 'long-run forcing' or **I(1) weakly exogenous** with respect to the parameters of the VECMX* model
- The VECMX* models are estimated separately for each country conditional on $\mathbf{x}_{it}^*$, using reduced rank regression, taking into account the possibility of cointegration both within $\mathbf{x}_{it}$ and across $\mathbf{x}_{it}$ and $\mathbf{x}_{it}^*$.
- This way, the number of cointegrating relations, r_i, the speed of adjustment coeffifients, α_i, and the cointegrating vectors β_i for each country model are obtained.

Solving the GVAR model

- Although estimation is done on a country by country basis, the GVAR model is solved for the world as a whole (in terms of a $k \times 1$ global variable vector, $k = \sum_{i=0}^{N} k_i$), ...
- ... taking account of the fact that all the variables are endogenous to the system as a whole.
- Specifically, after estimating the individual country VECMX* models, the corresponding VARX* models are recovered. Starting from the estimated country-specific VARX* (p_i, q_i) models:

$$\begin{aligned} \mathbf{x}_{it} =& \mathbf{a}_{i0} + \mathbf{a}_{i1} t + \mathbf{\Phi}_{i1} \mathbf{x}_{i,t-1} + \ldots + \mathbf{\Phi}_{ip_i} \mathbf{x}_{i,t-p_i} + \mathbf{\Lambda}_{i0} \mathbf{x}^*_{it} \\ &+ \mathbf{\Lambda}_{i1} \mathbf{x}^*_{i,t-1} + \ldots + \mathbf{\Lambda}_{iq_i} \mathbf{x}^*_{i,t-q_i} + \mathbf{u}_{it} \end{aligned}$$

Solving the GVAR model

- define $\mathbf{z}_{it}$ by $\mathbf{z}_{it} = \begin{pmatrix} \mathbf{x}_{it} \\ \mathbf{x}_{it}^{*} \end{pmatrix}$

- Assuming that $p_i = q_i$ for ease of exposition, write the VARX* (p_i, q_i) model for each economy as

$$\mathbf{A}_{i0}\mathbf{z}_{it} = \mathbf{a}_{i0} + \mathbf{a}_{i1}t + \mathbf{A}_{i1}\mathbf{z}_{it-1} + \ldots + \mathbf{A}_{ip_i}\mathbf{z}_{it-p_i} + \mathbf{u}_{it},$$

where $\mathbf{A}_{i0} = (\mathbf{I}_{k_i}, -\mathbf{\Lambda}_{i0}), \quad \mathbf{A}_{ij} = (\mathbf{\Phi}_{ij}, \mathbf{\Lambda}_{ij})$ for $j = 1, \ldots, p_i$.

- We can then use the so called link matrices $\mathbf{W}_i$, defined by the weights w_{ij}, to obtain the identity: $\mathbf{z}_{it} = \mathbf{W}_i\mathbf{x}_t$,

- where $\mathbf{x}_t = (\mathbf{x}_{0t}', \mathbf{x}_{1t}', \ldots, \mathbf{x}_{Nt}')'$ is the $k \times 1$ vector which collects all the endogenous variables of the system, and $\mathbf{W}_i$ is a $(k_i + k_i^{*}) \times k$ matrix.

Solving the GVAR model

- Using the identity $\mathbf{z}_{it} = \mathbf{W}_i \mathbf{x}_t$, it follows that

$$\mathbf{A}_{i0}\mathbf{W}_i\mathbf{x}_t = \mathbf{a}_{i0} + \mathbf{a}_{i1}t + \mathbf{A}_{i1}\mathbf{W}_i\mathbf{x}_{t-1} + \ldots + \mathbf{A}_{ip_i}\mathbf{W}_i\mathbf{x}_{t-p_i} + \mathbf{u}_{it}, \text{ for } i = 0, 1, 2, \ldots, N,$$

- and these individual models are then stacked to yield the model for $\mathbf{x}_t$ given by

$$\mathbf{G}_0\mathbf{x}_t = \mathbf{a}_0 + \mathbf{a}_1\mathbf{t} + \mathbf{G}_1\mathbf{x}_{t-1} + \ldots + \mathbf{G}_p\mathbf{x}_{t-p} + \mathbf{u}_t$$

- where

$$\mathbf{G}_0 = \begin{pmatrix} \mathbf{A}_{00}\mathbf{W}_0 \\ \mathbf{A}_{10}\mathbf{W}_1 \\ \vdots \\ \mathbf{A}_{N0}\mathbf{W}_N \end{pmatrix}, \mathbf{G}_j = \begin{pmatrix} \mathbf{A}_{0j}\mathbf{W}_0 \\ \mathbf{A}_{1j}\mathbf{W}_1 \\ \vdots \\ \mathbf{A}_{Nj}\mathbf{W}_N \end{pmatrix} \text{ for } j = 1, \ldots, p,$$

Solving the GVAR model

$$\mathbf{a}_0 = \begin{pmatrix} \mathbf{a}_{00} \\ \mathbf{a}_{10} \\ \vdots \\ \mathbf{a}_{N0} \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} \mathbf{a}_{01} \\ \mathbf{a}_{11} \\ \vdots \\ \mathbf{a}_{N1} \end{pmatrix}, \mathbf{u}_t = \begin{pmatrix} \mathbf{u}_{0t} \\ \mathbf{u}_{1t} \\ \vdots \\ \mathbf{u}_{Nt} \end{pmatrix}$$

and $p = \max p_i$ across all i. In general $p = \max(\max p_i, \max q_i)$.

- Since $\mathbf{G}_0$ is a known non-singular matrix that depends on the weights and parameter estimates, premultiplying the stacked model by $\mathbf{G}_0^{-1}$, the GVAR(p) model is obtained as

$$\mathbf{x}_t = \mathbf{b}_0 + \mathbf{b}_1\mathbf{t} + \mathbf{F}_1\mathbf{x}_{t-1} + \ldots + \mathbf{F}_p\mathbf{x}_{t-p} + \varepsilon_t,$$

where

$$\mathbf{b}_0 = \mathbf{G}_0^{-1}\mathbf{a}_0, \mathbf{b}_1 = \mathbf{G}_0^{-1}\mathbf{a}_1,$$
$$\mathbf{F}_j = \mathbf{G}_0^{-1}\mathbf{G}_j, \quad j = 1, \ldots, p, \varepsilon_t = \mathbf{G}_0^{-1}\mathbf{u}_t.$$

Solving the GVAR model

- The GVAR(p) can be solved recursively and used for a variety of purposes.
- There are no restrictions placed on the covariance matrix $\Sigma_\varepsilon = \mathbf{E}\left(\varepsilon_t \varepsilon_t'\right)$, unless one specifically decides to do so.
- The initial VARX$^*\left(p_i, q_i\right)$ model can also be extended to include common factors representing global variables, $\mathbf{d}_t$, such as oil prices. Wweak exogeneity is also assumed for global variables.

Part 2:

Theoretical Justification of the GVAR Approach

Based on Chudik and Pesaran (2016)

Approximation to a Global Factor Model

- A first attempt at a theoretical justification of the GVAR approach was provided by Dees et al. (2007).
- The authors derived the VARX* (p_i, q_i) model as an approximation to a global factor model.
- Their starting point is the following canonical global factor model (abstracting from deterministic terms and observed factors):

$$\mathbf{x}_{it} = \boldsymbol{\Gamma}_i \mathbf{f}_t + \xi_{it}, \quad \text{for } i = 1, 2, \ldots, N \tag{1}$$

- For each $i, \boldsymbol{\Gamma}_i$ is a $k_i \times m$ matrix of factor loadings, assumed to be uniformly bounded ($\|\boldsymbol{\Gamma}_i\| < K < \infty$), and ξ_{it} is a $k_i \times 1$ vector of country-specific effects.

Approximation to a Global Factor Model

- Factors and the country effects are assumed to satisfy

$$\mathbf{\Delta f}_t = \mathbf{\Lambda}_f(L)\eta_{ft}, \eta_{ft} \sim \text{IID}\left(\mathbf{0}, \mathbf{I}_m\right) \tag{2}$$

$$\mathbf{\Delta}\xi_{it} = \mathbf{\Xi}_i(L)\mathbf{u}_{it}, \mathbf{u}_{it} \sim \text{IID}\left(\mathbf{0}, \mathbf{I}_{k_i}\right), \quad \text{for } i = 1, 2, \ldots, N \tag{3}$$

- where $\mathbf{\Lambda}_f(L) = \sum_{\ell=0}^{\infty} \mathbf{\Lambda}_{f\ell} L^{\ell}$, $\mathbf{\Xi}_i(L) = \sum_{\ell=0}^{\infty} \mathbf{\Xi}_{i\ell} L^{\ell}$, and the coefficient matrices $\mathbf{\Lambda}_{f\ell}$ and $\mathbf{\Xi}_{i\ell}$, for $i = 1, 2, \ldots, N$, are uniformly absolute summable, which ensures the existence of $\text{Var}\left(\mathbf{\Delta f}_t\right)$ and $\text{Var}\left(\mathbf{\Delta}\xi_{it}\right)$. In addition, $\left[\mathbf{\Xi}_i(L)\right]^{-1}$ is assumed to exist.

- Under these assumptions, after first differencing (1) and using (3), they obtain:

$$\left[\mathbf{\Xi}_i(L)\right]^{-1}(1-L)\left(\mathbf{x}_{it} - \mathbf{\Gamma}_i\mathbf{f}_t\right) = \mathbf{u}_{it} \tag{4}$$

Approximation to a Global Factor Model

- Using the approximation:

$$(1-L)\left[\boldsymbol{\Xi}_i(L)\right]^{-1} \approx \sum_{\ell=0}^{p_i} \boldsymbol{\Phi}_{i\ell} L^{\ell} = \boldsymbol{\Phi}_i\left(L, p_i\right), \tag{5}$$

  they further obtain the following approximate VAR(p_i) model with factors:

$$\boldsymbol{\Phi}_i\left(L, p_i\right)\mathbf{x}_{it} \approx \boldsymbol{\Phi}_i\left(L, p_i\right)\boldsymbol{\Gamma}_i\mathbf{f}_t + \mathbf{u}_{it} \tag{6}$$

  for $i = 1, 2, \ldots, N$. Note that lags of other units do not feature in (6), and the errors, $\mathbf{u}_{it}$, are assumed to be cross-sectionally independently distributed.

- Unobserved common factors in (6) can be estimated by linear combinations of cross-section averages of observable variables, $\mathbf{x}_{it}$.

Approximation to a Global Factor Model

- Let $\tilde{\mathbf{W}}_i$ be the $k \times k^*$ matrix of country-specific weights and assume that it satisfies the granularity conditions given by:

$$\left\| \tilde{\mathbf{W}}_i \right\| < K N^{-\frac{1}{2}}, \quad \text{for all } i \tag{7}$$

$$\frac{\left\| \tilde{\mathbf{W}}_{ij} \right\|}{\left\| \tilde{\mathbf{W}}_i \right\|} < K N^{-\frac{1}{2}}, \quad \text{for all } i, j \tag{8}$$

- where $\tilde{\mathbf{W}}_{ij}$ are the blocks in the partitioned form of $\tilde{\mathbf{W}}_i = \left(\tilde{\mathbf{W}}'_{i1}, \tilde{\mathbf{W}}'_{i2}, \ldots, \tilde{\mathbf{W}}'_{iN} \right)'$, and the constant $K < \infty$ does not depend on i, j or N.

Approximation to a Global Factor Model

- Taking cross-section averages of $\mathbf{x}_{it}$ given by (1) yields

$$\mathbf{x}_{it}^{*} = \tilde{\mathbf{W}}_i'\mathbf{x}_t = \Gamma_i^{*}\mathbf{f}_t + \xi_{it}^{*} \tag{9}$$

- where $\|\mathbf{\Gamma}_i^{*}\| = \left\|\tilde{\mathbf{W}}_i'\mathbf{\Gamma}\right\| \leq \left\|\tilde{\mathbf{W}}_i'\right\| \|\mathbf{\Gamma}\| < K, \mathbf{\Gamma} = \left(\mathbf{\Gamma}_1', \mathbf{\Gamma}_2', \ldots, \mathbf{\Gamma}_N'\right)'$, and ξ_{it}^{*} satisfies

$$\mathbf{\Delta}\xi_{it}^{*} = \sum_{j=1}^{N} \tilde{\mathbf{W}}_{ij}'\mathbf{\Delta}\xi_{it} = \sum_{j=1}^{N} \tilde{\mathbf{W}}_{ij}'\mathbf{\Xi}_i(L)\mathbf{u}_{it}$$

- Assuming that $\mathbf{\Delta}\xi_{it}, i = 1, 2, ..., N$, are covariance stationary and weakly cross-sectionally dependent, they show that for each $t, \mathbf{\Delta}\xi_{it}^{*} \overset{q.m.}{\to} \mathbf{0}$ as $N \to \infty$, which implies $\xi_{it}^{*} \overset{q.m.}{\to} \xi_i^{*}$.

Approximation to a Global Factor Model

- It follows that under the additional condition that $\boldsymbol{\Gamma}_i^*$ has a full column rank,

$$\mathbf{f}_t \overset{q.m.}{\rightarrow} \left(\boldsymbol{\Gamma}_i^{*\prime}\boldsymbol{\Gamma}_i^*\right)^{-1}\boldsymbol{\Gamma}_i^*\left(\mathbf{x}_{it}^* - \xi_i^*\right)$$

as $N \to \infty$, which justifies using $(1, \mathbf{x}_{it}^*)'$ as proxies for the unobserved common factors.

- Thus, for N sufficiently large, they obtain the following country-specific VAR models augmented with $\mathbf{x}_{it}^*$:

$$\boldsymbol{\Phi}_i\left(L,\ p_i\right)\left(\mathbf{x}_{it} - \widetilde{\delta}_i - \widetilde{\boldsymbol{\Gamma}}_i\mathbf{x}_{it}^*\right) \approx \mathbf{u}_{it} \tag{10}$$

- where $\widetilde{\delta}_i$ and $\widetilde{\boldsymbol{\Gamma}}_i$ are given in terms of ξ_i^* and $\boldsymbol{\Gamma}_i^*$. Equation (10) motivates the use of VARX* conditional country models as an approximation to a global factor model.

Approximating Factor-Augmented Stationary High Dimensional VARs

- Chudik and Pesaran (2011) consider the conditions on the unknown parameters of a high-dimensional VAR model that would deliver individual VARX* models when N is large.

- In particular, they consider the following factor augmented high-dimensional VAR model:

$$(\mathbf{x}_t - \mathbf{\Gamma}\mathbf{f}_t) = \mathbf{\Theta}(\mathbf{x}_{t-1} - \mathbf{\Gamma}\mathbf{f}_{t-1}) + \mathbf{u}_t \tag{11}$$

- where $\mathbf{x}_t$ is $k \times 1$ vector of endogenous variables, $\mathbf{\Gamma}$ is a $k \times m$ matrix of factor loadings and $\mathbf{f}_t$ is an $m \times 1$ covariance stationary process of unobserved common factors.

- To simplify the exposition, the lag order (p) is set to unity.

Approximating Factor-Augmented Stationary High Dimensional VARs

- The authors assume that $\varrho(\mathbf{\Theta\Theta}') < 1 - \epsilon$, where $\epsilon > 0$ is an arbitrary small constant that does not depend on N, and $\mathbf{u}_t$ is weakly cross-sectionally dependent such that $\|E(\mathbf{u}_t\mathbf{u}_t')\| = \|\mathbf{\Sigma}_u\| < K$.
- The condition that the spectral radius of $\mathbf{\Theta\Theta}'$ is below and bounded away from unity is a slightly stronger requirement than the usual stationarity condition that assumes the eigenvalues of $\mathbf{\Theta}$ lie within the unit circle.
- The stronger condition is needed to ensure that variances exist when $N \to \infty$ (see an illustrative example in Chudik and Pesaran (2016)).

Approximating Factor-Augmented Stationary High Dimensional VARs

- Similarly, as in Dees et al. (2007), it is assumed in (11) that factors are included in the VAR model in an additive way so that $\mathbf{x}_t$ can be written as:

$$\mathbf{x}_t = \mathbf{\Gamma f}_t + \xi_t \tag{12}$$

- where $\xi_t = (\mathbf{I}_k - \mathbf{\Theta L})^{-1}\mathbf{u}_t$, and the existence of the inverse of $(\mathbf{I}_k - \mathbf{\Theta L})$ is ensured by the assumption on $\varrho(\mathbf{\Theta\Theta}')$ above. One could also consider the alternative factor augmentation setup:

$$\mathbf{x}_t = \mathbf{\Theta x}_{t-1} + \mathbf{\Gamma f}_t + \mathbf{u}_t \tag{13}$$

- where factors are added to the errors of the VAR model, instead of (11), where deviations of $\mathbf{x}_t$ from the factors are modeled as a VAR.

Approximating Factor-Augmented Stationary High Dimensional VARs

- For any set of weights represented by the $k \times k^*$ matrix $\tilde{\mathbf{W}}_\mathbf{i}$, we obtain (using (12))

$$\mathbf{x}_{it}^* = \tilde{\mathbf{W}}_i' \mathbf{x}_t = \mathbf{\Gamma}_i^* \mathbf{f}_t + \xi_{it}^*$$

  where $\mathbf{\Gamma}_i^* = \tilde{\mathbf{W}}_i' \mathbf{\Gamma}$ and

$$\xi_{it}^* = \tilde{\mathbf{W}}_i' (\mathbf{I_k} - \mathbf{\Theta L})^{-1} \mathbf{u}_t$$

- Chudik and Pesaran (2011) show that if $\tilde{\mathbf{W}}_\mathbf{i}$ satisfies (7), then

$$\begin{aligned} \left\| E(\xi_{it}^* \xi_{it}^{*\prime}) \right\| &= \left\| \sum_{\ell=0}^{\infty} \tilde{\mathbf{W}}_i \mathbf{\Theta}^\ell E(\mathbf{u}_{t-\ell} \mathbf{u}_{t-\ell}') \mathbf{\Theta}^\ell \tilde{\mathbf{W}}_i \right\| \\ &\leq \left\| \tilde{\mathbf{W}}_i \right\|^2 \left\| \Sigma_u \right\| \sum_{\ell=0}^{\infty} \left\| \mathbf{\Theta}^\ell \right\|^2 \\ &= O(N^{-1}) \end{aligned} \tag{14}$$

Approximating Factor-Augmented Stationary High Dimensional VARs

- where $\left\| \tilde{\mathbf{W}}_i \right\|^2 = O(N^{-1})$ by (7), $\|\Sigma_u\| < K$ by the weak error $(\mathbf{u}_t)$ cross-section dependence assumption and $\sum_{\ell=0}^{\infty} \|\boldsymbol{\Theta}^{\ell}\|^2 < K$ by the assumption on the spectral radius of $\varrho(\boldsymbol{\Theta}\boldsymbol{\Theta}')$.

- Equation (14) establishes that $\xi_{it}^* \overset{q.m.}{\rightarrow} \mathbf{0}$ (uniformly in i and t) as $N, T \overset{j}{\rightarrow} \infty$. It now follows that

$$\mathbf{x}_{it}^* - \boldsymbol{\Gamma}_i^* \mathbf{f}_t \overset{q.m.}{\rightarrow} \mathbf{0}, \text{as } N, T \overset{j}{\rightarrow} \infty \tag{15}$$

- which confirms the well-known result that only strong cross-section dependence can survive large N aggregation with granular weights.

Approximating Factor-Augmented Stationary High Dimensional VARs

- Therefore, the unobserved common factors can be approximated by cross-section averages $\mathbf{x}_t^*$ in this dynamic setting, provided that $\boldsymbol{\Gamma}_i^*$ has full column rank.

- Now it is easy to see what additional requirements are needed on the coefficient matrix $\boldsymbol{\Theta}$ to obtain country VARX* models when N is large.

- The model for the country-specific variables, $\mathbf{x}_{it}$, from the system (11) is given by:

$$\mathbf{x}_{it} = \boldsymbol{\Theta}_{ii}\mathbf{x}_{it-1} + \sum_{j=1, j\neq i} \boldsymbol{\Theta}_{ij}\left(\mathbf{x}_{j,t-1} - \boldsymbol{\Gamma}_j \mathbf{f}_t\right) + \boldsymbol{\Gamma}_i \mathbf{f}_t - \boldsymbol{\Theta}_i' \boldsymbol{\Gamma}_i f_{t-1} + \mathbf{u}_{it} \quad (16)$$

Approximating Factor-Augmented Stationary High Dimensional VARs

where $\mathbf{\Theta}_{ij}$ are appropriate partitioned submatrices of

$$\mathbf{\Theta} = \begin{pmatrix} \mathbf{\Theta}_{11} & \mathbf{\Theta}_{12} & \cdots & \mathbf{\Theta}_{1N} \\ \mathbf{\Theta}_{21} & \mathbf{\Theta}_{22} & & \mathbf{\Theta}_{2N} \\ \vdots & & \ddots & \vdots \\ \mathbf{\Theta}_{N1} & \mathbf{\Theta}_{N2} & \cdots & \mathbf{\Theta}_{NN} \end{pmatrix}$$

- Suppose now that $\|\mathbf{\Theta}_{ij}\| < \frac{K}{N}$, for all $i \neq j$
- This assumption implies that the matrix $\mathbf{\Theta}_{-i} = (\mathbf{\Theta}_{i1}, \mathbf{\Theta}_{i2}, \ldots, \mathbf{\Theta}_{i,i-1}, 0, \mathbf{\Theta}_{i,i+1}, \ldots, \mathbf{\Theta}_{iN})'$ satisfies the granularity condition (7), in particular $\|\mathbf{\Theta}_{-i}\|^2 < KN^{-1}$, and using (14) but with $\mathbf{\Theta}_{-i}$ instead of $\tilde{\mathbf{W}}_i$, we obtain:

Approximating Factor-Augmented Stationary High Dimensional VARs

$$\sum_{j=1, j\neq i} \boldsymbol{\Theta}_{ij} \left(\mathbf{x}_{j,t-1} - \boldsymbol{\Gamma}_j \mathbf{f}_t \right) \overset{q.m.}{\to} \mathbf{0} \text{ as } N \to \infty \tag{17}$$

- Finally, substituting (15) and (17) in (16) we obtain the country-specific VARX*(1, 1) model

$$\mathbf{x}_{it} - \boldsymbol{\Theta}_{ii}\mathbf{x}_{it-1} - \boldsymbol{\Lambda}_{i0}\mathbf{x}_t^* - \boldsymbol{\Lambda}_{i1}\mathbf{x}_{t-1}^* - \mathbf{u}_{it} \overset{q.m.}{\to} \mathbf{0} \\ \text{uniformly in } i\text{, and as } N \to \infty \tag{18}$$

where $\boldsymbol{\Lambda}_{i0} = \boldsymbol{\Gamma}_i \left(\boldsymbol{\Gamma}^{*\prime} \boldsymbol{\Gamma}^* \right)^{-1} \boldsymbol{\Gamma}^*$, and $\boldsymbol{\Lambda}_{i1} = \boldsymbol{\Theta}_i' \boldsymbol{\Gamma}_i \left(\boldsymbol{\Gamma}^{*\prime} \boldsymbol{\Gamma}^* \right)^{-1} \boldsymbol{\Gamma}^*$

- The requirement $\|\boldsymbol{\Theta}_{ij}\| < \frac{K}{N}$, for all $i \neq j$, with the remaining assumptions are thus sufficient to obtain the VARX$^*(p_i, q_i)$ models when N is sufficiently large.

Part 3:

Additional Econometric Considerations

Please refer to the Econometrics Appendix
of the GVAR User Guide for mathematical details

Specification Tests

Before solving the GVAR model, a number of tests can be performed to ensure that the model is well-specified and that the assumptions (broadly) hold. Those include:

- **Unit root tests**: to test the integration properties of all variables (domestic, foreign, and global). This can be done using standard ADF unit root t-statistics, or those based on weighted symmetric estimation of ADF type regressions, among others.
- **Residual serial correlation tests**: The residuals of the individual VARX* models are assumed to be serially uncorrelated. This can be formally tested using an F-version of the familiar Lagrange Multiplier (LM) test.

Specification Tests

- **Tests of Co-trending restrictions**: to test of whether the cointegrating relations are trended. This can be done using likelihood-ratio tests.
- **Tests of overidentifying restrictions on the cointegrating vectors**: One may wish to incorporate long-run structural relationships to develop a global model with a theoretically coherent foundation. This can be done by imposing overidentying restrictions, which can be tested for using likelihood-ratio tests.
- **Weak Exogeneity Tests**: The weak exogeneity assumption can be formally tested via a test of the joint significance of the estimated error correction terms in auxiliary equations for the country-specific foreign variables

Impulse Response Analysis

- Impulse responses refer to the time profile of the effects of **variable-specific shocks** or **identified shocks** on the future states of a dynamical system and thus, on all the variables in the model.
- The impulse responses of shocks to specific variables considered for the GVAR model are the **generalized impulse response functions (GIRFs)**
- **Structural generalized impulse response functions (SGIRFs)** and **orthogonalized impulse response functions (OIRFs)** can also be computed for shocks identified to a single country and to all countries, respectively.

Forecast Error Variance Decomposition

- Traditionally the forecast error variance decomposition of a VAR model is performed on a set of orthogonalized shocks, whereby the contributions of the orthogonalized innovations to the mean square errors of the model's forecasts are calculated.
- In the GVAR model, the shocks are not orthogonal, so an alternative approach is to compute a **generalized forecast error variance decomposition (GFEVD)** that is invariant to the ordering of the variables.
- Similarly to impulse responses, In the cases of structural identification to a single country and to all countries, one can perform **structural generalized forecast error variance decomposition (SGFEVD)** and **orthogonalized forecast error varaiance decomposition (OFEVD)** respectively.

Next session:

Computer Lab: Specification and Estimation of a GVAR Model

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