

A Structural Foundation for Trade Complementarity

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Abstract

Two countries' trade structures are complementary when what one sells is what the other wants to buy. Complementarity in this sense is conventionally measured by indices computed from realized trade, which leaves such measures without a structural interpretation and unable to explain the flows from which they are built. This paper derives a complementarity index from model primitives in a multi-sector Eaton and Kortum (2002) economy. The Structural Complementarity Index pairs an exporter's technology-driven comparative-advantage profile with an importer's demand composition, each recovered from a single country's fundamentals and neither from trade within the pair, so the index cannot be recovered from the bilateral flow it is meant to explain. We then relate it to the Trade Complementarity Index of Michaely (1996), one of the most widely used descriptive measures. The two apply different functionals, a sum of products and a sum of minima, to arguments that, in general, differ. When home trade shares are uniform across sectors and each exporter is small relative to world sector size, those arguments coincide, and the descriptive index bounds the structural one from above, up to a remainder we bound explicitly. The bound binds with equality exactly under complete Ricardian specialization, the deterministic limit of the model. When trade costs and importers' supplier access do not vary across sectors, the model yields a structural gravity equation with the Structural Complementarity Index as a regressor, which provides a baseline empirical specification for estimating the effect of complementarity on bilateral trade.

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1. Introduction

Two countries trade heavily when what one sells is what the other wants to buy. The intuition long predates any formal trade model, and the Trade Complementarity Index of [Michaely \(1996\)](#)¹ operationalizes it directly. One compares country n 's export composition across sectors with country i 's import composition and scores the overlap. The index is simple, transparent, and widely used in policy work on regional integration, including the sizable literature on the African Continental Free Trade Area (AfCFTA). It is also atheoretic by construction. The composition vectors it compares are themselves equilibrium outcomes, shaped by technology and by the trade costs each country faces against the world, yet the index takes them as given and models neither force. It can therefore register a low score without indicating whether the two countries produce and consume genuinely mismatched baskets or whether costs are what hold their trade down.

We propose a structural alternative. The environment is the multi-sector Eaton–Kortum economy of [Eaton and Kortum \(2002\)](#), in the sectoral form developed by [Costinot, Donaldson and Komunjer \(2012\)](#) and employed throughout the quantitative-trade literature that follows [Caliendo and Parro \(2015\)](#). Within that environment we construct a Structural Complementarity Index (SCI) from two objects, each of which belongs to a single country: the exporter's technology-driven comparative-advantage profile across sectors, and the importer's demand composition across those same sectors. Neither object references the trade relationship between the exporter and importer being scored.

The restriction is deliberate and not merely aesthetic. A natural first attempt at a structural index would use the bilateral trade share that the model itself delivers. That object, however, turns out to be algebraically identical to realized bilateral trade divided by importer spending, so any regression of trade on it is tautological. Building the index from one-sided fundamentals avoids the problem at the level of the definition rather than repairing it afterward, and it leaves SCI with the shares-and-shocks structure for which the shift-share literature has developed identification results ([Borusyak, Hull and Jaravel, 2022](#); [Adão, Kolesár and Morales, 2019](#)).

SCI is deliberately silent on trade costs. The exporter's profile is evaluated at world competitiveness, a function of technology and wages alone, and the importer's demand composition is a preference parameter. In this respect SCI resembles the descriptive index it is meant to improve upon, and the improvement lies elsewhere. Where the Michaely index consumes realized shares in which technology and costs are already mixed together and cannot afterward be separated, SCI holds costs out by construction, which is what makes it informative about them. Measured against realized trade, a cost-free benchmark reveals what geography contributes. Proposition 2 pins down what trade would be were costs absent, and Proposition 1 gives the exact wedge that costs drive between bilateral and world competitiveness. The two do not collapse into a single multiplicative

¹[Michaely \(1996\)](#) is a World Bank Policy Research Working Paper, not a journal article; we cite it because it is the paper the index is conventionally attributed to in the literature that uses it. An algebraically related but conceptually distinct measure is the export-similarity index of [Finger and Kreinin \(1979\)](#), published in the *Economic Journal*: it applies the same sum-of-minima operator, but to two *exporters'* export compositions, to gauge how much one country's export mix competes with another's in third markets, rather than to one country's exports against a second country's imports. We do not use it here.

geography term, since that wedge varies across sectors and depends on the importer’s own position in each, but together they state precisely how far realized trade departs from what complementarity alone would imply.

A second question follows naturally once SCI is in hand. Does the model deliver an object that the Michaely index approximates, and if so under what conditions and with what error? The answer is not an algebraic identity, since the two indices apply different functionals, a bilinear sum of products in one case and a sum of minima in the other, to arguments that in general differ. Two conditions nonetheless align the arguments. If home trade shares are uniform across sectors, and if each exporter’s own absorption is small relative to world sector size, then SCI and the Michaely index become functionals of one and the same pair of vectors. Lemma 4 establishes a sharp inequality between those two functionals, and Theorem 1 converts it into the bound $\text{SCI}_{ni} \leq \text{TCI}_{ni}$, up to a remainder that we bound explicitly. The inequality binds with equality precisely when the exporter or the importer is completely specialized sector by sector, which is the deterministic Ricardian limit of the model as the Fréchet dispersion parameter θ grows without bound.

Realized trade enters through a separate channel. Under frictionless trade costs, and without any appeal to the small-exporter idealization, bilateral trade relative to importer spending equals $\Omega_n \text{SCI}_{ni}$ exactly, where Ω_n denotes the exporter’s aggregate world competitiveness (Proposition 2). Complementarity therefore does not pin down realized trade on its own. Two exporters whose sectoral profiles match an importer equally well but whose overall competitiveness differs will trade with that importer at different volumes. Once geography is switched off, however, a single scale factor is all that separates the two magnitudes.

Beyond the frictionless benchmark, trade costs that vary across country pairs can be accommodated. When trade costs and importers’ supplier access do not vary across sectors, the model yields a structural gravity equation in which $\ln \text{SCI}_{ni}$ enters with a unit coefficient alongside exporter and importer fixed effects (Proposition 4). This equation provides a baseline empirical specification for estimating the effect of complementarity on bilateral trade.

The paper proceeds as follows. Section 1.1 situates the argument in four literatures. Section 2 sets out the model and establishes an invariance property of demand composition under Cobb–Douglas preferences that the construction in Section 8 later relies on. Section 3 defines both indices along with the exporter’s structural competitiveness profile, and records the sum-of-minima representation of the Michaely index. Section 4 derives an exact formula for the gap between bilateral and world-level competitiveness. Section 5 contains the two central results, the exact relationship to realized trade and the nesting theorem. Section 6 derives a structural gravity equation with SCI_{ni} as a regressor and, from it, a baseline empirical specification for estimating the effect of complementarity on bilateral trade. Sections 7 and 8 take up identification and measurement. Section 9 works through a numerical example in which every proposition, the theorem, and the bound are checked against hand computation. Section 10 discusses non-homothetic demand and Section 11 concludes. Appendix A derives the trade-share equation from primitives and Appendix B collects the proofs.

1.1 Related Literature

The argument draws on four literatures.

(i) *Complementarity and Trade-Intensity Indices*. Michaely’s index (Michaely, 1996) belongs to an older family of descriptive measures that summarize bilateral trade patterns from raw flow data without appeal to a model. Two close relatives apply the same sum-of-minima operator to different objects. The export-similarity index of Finger and Kreinin (1979) compares two exporters’ compositions with each other rather than an exporter’s with an importer’s, while the trade-intensity index surveyed by Drysdale and Garnaut (1982) asks whether a bilateral pair trades more or less than a gravity-neutral benchmark would predict, using trade values rather than composition shares. All three remain in routine use in applied work on regional integration, largely because they require nothing beyond bilateral trade data: no estimated elasticities, no assumed cost structure, no closed model. Their portability is also their limitation. A pair may score low on the Michaely index because their production structures are poorly matched, or because the costs each faces against the world have pushed their trade into different sectors. The index cannot separate the two explanations, since it inherits both influences through the realized shares it consumes and models neither of them. We are not aware of a prior derivation that embeds the index in the primitives of a quantitative trade model and asks what it recovers. The closest antecedents supply gravity-consistent intensity benchmarks for the comparison of trade values (Drysdale and Garnaut, 1982) rather than a structural counterpart to the composition-based index, which is the object derived here.

(ii) *Quantitative Ricardian Trade*. The multi-sector Eaton–Kortum structure originates with Eaton and Kortum (2002) and was formalized in its now-standard sectoral form by Costinot, Donaldson and Komunjer (2012), who show how the Fréchet machinery delivers a Ricardian mapping from technology to the pattern of specialization. Caliendo and Parro (2015) embed the same structure in a general-equilibrium model with input-output linkages to quantify the trade and welfare effects of NAFTA, and we follow their treatment of the trade elasticity θ , or its sector-specific analogue θ^s , in applied work. Arkolakis, Costinot and Rodríguez-Clare (2012) show that a broad class of quantitative trade models, Eaton–Kortum among them, share a common sufficient-statistics formula for the gains from trade, which underscores how much applied content rests on the trade-share object in equation (1). Costinot and Rodríguez-Clare (2014) survey the wider toolkit these models support. The contribution here is not a new model, since every primitive in Section 2 is standard, but a mapping from the model’s reduced form to an empirical object the literature has not previously targeted.

(iii) *Market Access and Economic Geography*. The term Φ_i^s in equation (1) is a sector-level market-access object of the kind Donaldson and Hornbeck (2016) use to value nineteenth-century American railroad expansion. Redding and Turner (2015) survey the broader reduced-form and structural toolkit connecting transportation costs to the spatial organization of economic activity. The relationship established in Section 5.1 between SCI_{ni} and realized bilateral trade under frictionless costs is in effect a statement about how complementarity and market access combine, so one framework can address both the composition question taken up here and the market-access question

of what cheaper entry into a given destination would be worth to an exporter.

(iv) *Shift-Share Identification and Estimation.* The shares-and-shocks structure of SCI_{ni} documented in Section 7 places its identification within the shift-share framework of [Borusyak, Hull and Jaravel \(2022\)](#) and [Adão, Kolesár and Morales \(2019\)](#), who establish conditions for exogeneity from two complementary directions, quasi-random shocks in the first case and quasi-random shares in the second. [Goldsmith-Pinkham, Sorkin and Swift \(2020\)](#) provide a widely used practitioner’s treatment of the same class of designs under the older Bartik label. We draw on this literature in Section 7 to argue that the predetermined-fundamentals structure of SCI_{ni} supports a shift-share identification argument, though we do not extend the framework itself. A full estimating equation, together with the pseudo-Poisson maximum-likelihood gravity estimator of [Santos Silva and Tenreyro \(2006\)](#) required for a panel that retains zeros, is left to a follow-up paper.

2. A Multi-Sector Eaton–Kortum Economy

The economy has N countries, indexed $n, i = 1, \dots, N$, and S sectors, indexed $s = 1, \dots, S$. Each sector contains a continuum of goods indexed by j , and every country is able to produce every good. Country n ’s efficiency at producing good j in sector s , written $z_n(j)$, is a random draw, independent across countries and goods, from a Fréchet distribution with scale (technology) parameter $T_n^s > 0$ and shape (trade elasticity) parameter $\theta > 1$, common across sectors for expositional simplicity; allowing sector-specific θ^s changes none of the arguments below. Shipping a unit of a sector- s good from n to i costs $d_{ni}^s \geq 1$ (iceberg), with $d_{nn}^s = 1$, and wages are $w_n > 0$. Standard EK aggregation gives i ’s expenditure share on n in sector s , *including* the domestic term $n = i$, as

$$\pi_{ni}^s = \frac{T_n^s (w_n d_{ni}^s)^{-\theta}}{\Phi_i^s}, \quad \Phi_i^s = \sum_{m=1}^N T_m^s (w_m d_{mi}^s)^{-\theta}, \quad \sum_{n=1}^N \pi_{ni}^s = 1. \quad (1)$$

This is the standard result of [Eaton and Kortum \(2002\)](#); Appendix A derives it in full, step by step, from the underlying Fréchet efficiency draws.

Country i spends $X_i^s > 0$ on sector s , measured as total absorption and sourced from anywhere including domestically, with $X_i \equiv \sum_s X_i^s$ and expenditure share $s_i^s \equiv X_i^s / X_i$. Following standard practice, we micro-found this allocation with a Cobb–Douglas upper tier, $U_i = \prod_s (C_i^s)^{\beta_i^s}$ with $\sum_s \beta_i^s = 1$, so that i divides income Y_i across sectors before choosing a source country within each sector. The choice of upper tier carries more weight than it might appear, because it yields an invariance property on which the measurement strategy of Section 8 depends.

Lemma 1 (Demand Composition Is Price-Invariant). *Under Cobb–Douglas upper-tier utility with weights β_i^s , expenditure on sector s is $X_i^s = \beta_i^s Y_i$ for any price vector, so $s_i^s = \beta_i^s$ exactly, independent of prices, trade costs, or which countries i trades with. (Proof in Appendix B.)*

Under Cobb–Douglas preferences the observed expenditure share does not simply proxy for the underlying taste parameter. The two coincide exactly, whatever trade costs and competitiveness

do to prices. This is what allows us to treat s_i^s as a trade-cost-independent primitive throughout, on the same footing as the technology parameters T_n^s on the supply side. We take $\{T_n^s, w_n, \beta_i^s, Y_i\}$ as given. Embedding them in a full general-equilibrium closure with labor-market clearing and balanced trade is standard (Caliendo and Parro, 2015) and immaterial for what follows, since the results concern the mapping from the trade-share equation (1) and the demand primitives to the complementarity indices rather than the equilibrium determination of wages and income.

Bilateral sector-level trade value is $X_{ni}^s = \pi_{ni}^s X_i^s$, and aggregate bilateral trade is

$$X_{ni} = \sum_s X_{ni}^s = \sum_s \pi_{ni}^s X_i^s = X_i \sum_s \pi_{ni}^s s_i^s. \quad (2)$$

3. Two Complementarity Indices

Equation (2) combines absorption shares s_i^s , which include what i buys from itself, with trade shares π_{ni}^s defined over all ordered pairs including $n = i$. The Michaely index, by contrast, is built entirely from cross-border flows. The two objects cannot be compared until that difference is made precise.

For $n \neq i$ define exports and imports net of domestic sales,

$$X_n^{\text{exp},s} \equiv \sum_{i \neq n} X_{ni}^s, \quad X_i^{\text{imp},s} \equiv \sum_{n \neq i} X_{ni}^s = X_i^s (1 - \pi_{ii}^s),$$

with totals $X_n^{\text{exp}} \equiv \sum_s X_n^{\text{exp},s}$ and $X_i^{\text{imp}} \equiv \sum_s X_i^{\text{imp},s}$, and composition vectors

$$x_n^s \equiv \frac{X_n^{\text{exp},s}}{X_n^{\text{exp}}}, \quad m_i^s \equiv \frac{X_i^{\text{imp},s}}{X_i^{\text{imp}}}.$$

The vectors x_n^s and m_i^s record what n actually exports and what i actually imports, shaped jointly by technology, trade costs, and history. The Michaely index compares them directly.

Definition 1 (Michaely TCI). $\text{TCI}_{ni} \equiv 1 - \frac{1}{2} \sum_s |x_n^s - m_i^s|$.

The index equals one minus half the total absolute distance between the two composition vectors, attaining one when they coincide and zero when their supports are disjoint.

Like the Michaely index, the structural alternative compares one-sided objects, each describing a single country against the world rather than the two against each other. What separates the two indices is where those objects come from. Michaely's are realized compositions, in which technology and costs are already blended; ours come from the primitives, and we read them off the trade-share equation rather than posit them. In (1), the exporter enters the numerator through the single product $T_n^s (w_n d_{ni}^s)^{-\theta}$, the denominator being a property of the destination market that aggregates all suppliers. That product factors into a term belonging to n alone, $T_n^s w_n^{-\theta}$, and a term belonging to the pair, $(d_{ni}^s)^{-\theta}$. Removing the second leaves the whole of n 's supply side and nothing else.

The surviving term is cost-adjusted technology. Producing one unit of good j in n costs $w_n/z_n(j)$, so a higher T_n^s , which shifts the distribution of efficiency draws upward, and a lower wage w_n both

push n 's production costs down and make it a more likely low-cost supplier. Appendix A makes the statement exact: the delivered-price distribution of country n is $\Pr[c_n^i \leq c] = 1 - \exp(-\phi_n^i c^\theta)$ with $\phi_n^i = T_n^s (w_n d_{ni}^s)^{-\theta}$, so a larger value of that term shifts n 's prices downward, and the trade elasticity θ governs how sharply a cost advantage converts into sales. Normalizing by the same quantity summed across all countries, which measures world supply capability in the sector, gives n 's world competitiveness

$$\omega_n^s \equiv \frac{T_n^s w_n^{-\theta}}{\Phi^s}, \quad \Phi^s \equiv \sum_m T_m^s w_m^{-\theta}, \quad (3)$$

with $\sum_n \omega_n^s = 1$, so that ω_n^s is a market-share vector across exporters carrying no trade cost and no importer-specific term. The interpretation is not merely suggestive. Proposition 1 below shows that when cross-border costs vanish, $\pi_{ni}^s = \omega_n^s$ in every destination at once, so ω_n^s is the market share n would command everywhere in a world without geography.

That object still measures absolute advantage, since a country with good technology across the board scores highly in every sector. Complementarity is a statement about comparative advantage instead, since whether what n produces suits what i wants should turn on which sectors n is relatively strong in and not on how strong n is overall. Normalizing ω_n^s by its sum across sectors, $\Omega_n \equiv \sum_s \omega_n^s$, separates the two: the level is carried entirely by Ω_n , and the normalized vector is unchanged whenever n 's competitiveness rises by the same proportion in every sector. What remains is n 's structural export composition

$$\hat{x}_n^s \equiv \frac{\omega_n^s}{\Omega_n}, \quad \sum_s \hat{x}_n^s = 1, \quad (4)$$

the sectoral mix n would export on the strength of technology alone, in a world without trade costs.

The importer's side is already in hand. By Lemma 1, s_i^s equals the preference weight β_i^s and so is a demand primitive, unaffected by prices or by trade costs. The two sides of the index therefore come from different primitives, technology on one and tastes on the other, and neither is a function of the pair (n, i) .

Definition 2 (Structural Complementarity Index). $\text{SCI}_{ni} \equiv \sum_s \hat{x}_n^s s_i^s$.

Because $\hat{x}_n$ and s_i are probability vectors over sectors, SCI_{ni} lies in $[0, 1]$, the same range as the Michaely index. Its construction uses no information about trade between n and i , and so it cannot be recovered from X_{ni} .

A second and subsidiary object records realized bilateral trade relative to importer spending,

$$G_{ni} \equiv \sum_s \pi_{ni}^s s_i^s = \frac{X_{ni}}{X_i}. \quad (5)$$

Here G_{ni} is the inner product of n 's bilateral competitiveness vector $\{\pi_{ni}^s\}_s$, which does depend on i through Φ_i^s , with i 's expenditure shares. Equation (2) makes it identically equal to realized trade normalized by importer spending. It is therefore a restatement of observed trade rather than a

measure of complementarity, and we do not treat it as one. It reappears only in Section 5.1, where we establish how it relates to SCI_{ni} once trade costs are specified.

The two indices apply different functionals to different arguments. Definition 1 applies a sum of minima to realized, cross-border composition shares; Definition 2 applies an inner product to structural objects built from technology and preferences. Closing the gap between them occupies Sections 4 and 5. The first step rewrites the Michaely index in the form that makes comparison possible.

Lemma 2 (Sum-of-Minima Identity). *For any two probability vectors $a, b \in \mathbb{R}_+^S$ with $\sum_s a_s = \sum_s b_s = 1$, $1 - \frac{1}{2} \sum_s |a_s - b_s| = \sum_s \min(a_s, b_s)$. Hence $\text{TCI}_{ni} = \sum_s \min(x_n^s, m_i^s)$.*

4. Exact Decomposition of Bilateral Competitiveness

Two gaps separate the indices. On the import side, the realized share m_i^s must be connected to the demand primitive s_i^s ; on the export side, the realized share x_n^s must be connected to the structural composition $\hat{x}_n^s$. The first gap closes with an exact identity, derived in Section 4.1. The second requires a formula relating bilateral competitiveness π_{ni}^s to world competitiveness ω_n^s , which Section 4.2 supplies with an explicit bound that vanishes in the frictionless limit.

4.1 Import Side: An Exact Identity

Lemma 1 has established that s_i^s equals the preference parameter β_i^s . What remains is whether it also equals m_i^s , the import-only composition from which the Michaely index is built. The obstacle is definitional rather than economic. The share s_i^s is measured against total absorption, imports and domestic purchases together, while m_i^s is measured against imports alone. Netting out domestic sourcing rescales every sector's spending by the same factor, and therefore leaves relative shares untouched, whenever that sourcing is equally prevalent across sectors.

Assumption 1 (Uniform Home Share). *For importer i , $\pi_{ii}^s = \pi_{ii}$ for all s (the domestic trade share does not vary across sectors for country i).*

Lemma 3 (Import Shares Equal Expenditure Shares). *Under Assumption 1, $m_i^s = s_i^s$ for every s , exactly.*

The result is an identity rather than an approximation, holding at any level of home bias $\pi_{ii} \in [0, 1)$ provided that level does not vary across sectors. Its role in what follows is narrow. It is required only for comparing SCI, which uses s_i^s , with the Michaely index, which uses m_i^s . It plays no part in constructing SCI_{ni} , which draws on s_i^s directly.

4.2 Export Side: An Exact Bias Formula

The obstacle on the exporter's side is substantive. Bilateral competitiveness π_{ni}^s differs across importers because Φ_i^s aggregates the suppliers competing for i 's market, whereas ω_n^s and $\hat{x}_n^s$ are

world objects that cannot vary by destination. The two coincide only if π_{ni}^s ceases to depend on i , which requires trade costs to be identical across destinations, so that no importer's local competitive environment makes n more or less competitive there than anywhere else.

Assumption 2 (Uniform Cross-Border Cost). *For sector s , $d_{ni}^s = \delta^s \geq 1$ for every ordered pair $n \neq i$ (the iceberg cost between any two distinct countries is the same, though it may vary by sector).*

Write $\kappa^s \equiv (\delta^s)^{-\theta} \in (0, 1]$ for the resulting cost factor.

Proposition 1 (Exact Bilateral Competitiveness under Uniform Cross-Border Cost). *Under Assumption 2, for every $n \neq i$,*

$$\pi_{ni}^s = \frac{\kappa^s \omega_n^s}{\kappa^s + (1 - \kappa^s) \omega_i^s}, \quad |\pi_{ni}^s - \omega_n^s| \leq \omega_n^s \omega_i^s \frac{1 - \kappa^s}{\kappa^s}. \quad (6)$$

In particular, if $\kappa^s = 1$ (i.e. $\delta^s = 1$, frictionless cross-border trade), then $\pi_{ni}^s = \omega_n^s$ exactly, for every $i \neq n$.

Equation (6) holds at any level of friction. The departure of bilateral competitiveness from its world-share benchmark is governed by two quantities, the importer's own world market share ω_i^s and the cost wedge $1 - \kappa^s$, and vanishes when either is zero. The second of these defines the benchmark case used in the remainder of the paper.

Assumption 3 (Frictionless Cross-Border Trade). *Assumption 2 holds with $\kappa^s = 1$ for every s .*

5. Structural Complementarity, Realized Trade, and TCI

The section divides in two. Under frictionless trade alone, Section 5.1 establishes an exact relationship between SCI_{ni} and realized bilateral trade, in which complementarity determines trade up to a single exporter-specific scale factor and nothing is approximated. Adding the small-exporter idealization, Section 5.2 then delivers an inequality between SCI_{ni} and the Michaely index, together with an error bound that vanishes as the idealization becomes exact.

5.1 The Exact Relationship to Realized Trade

When Assumption 3 holds, Proposition 1 yields $\pi_{ni}^s = \omega_n^s$ for every $i \neq n$, so bilateral competitiveness collapses onto the world-share benchmark. Realized trade G_{ni} from equation (5) can then be related directly to SCI_{ni} .

Proposition 2 (Realized Trade Equals Scaled Complementarity). *Under Assumptions 1 and 3, exactly,*

$$G_{ni} = \Omega_n \text{SCI}_{ni}.$$

The result requires no small-exporter idealization and holds at any country size. Realized bilateral trade, measured relative to importer spending, is complementarity scaled by the exporter's

aggregate world competitiveness Ω_n . Complementarity alone therefore does not determine G_{ni} , since two exporters matching i 's demand equally well but differing in overall competitiveness will generate different trade volumes. Away from the frictionless benchmark, G_{ni} also absorbs the bilateral cost d_{ni}^s , precisely the geographic content that a measure of complementarity ought to exclude. Proposition 2 is what makes SCI_{ni} usable on the right-hand side of a gravity regression for G_{ni} , or equivalently for X_{ni} , by separating complementarity from the scale factor multiplying it.

5.2 The Nesting Theorem

Connecting SCI_{ni} to the Michaely index requires knowing how far the realized export share x_n^s departs from its structural counterpart $\hat{x}_n^s$, since the former enters the index and the latter enters SCI_{ni} . Two further assumptions deliver a bound on that distance.

Assumption 4 (Equal Sector Size). $E^s \equiv \sum_i X_i^s = E$ for every sector s (world absorption is the same size in every sector).

Assumption 5 (ε -Small Exporter). $X_n^s \leq \varepsilon E$ for every s , for some $\varepsilon \in [0, 1)$ (country n 's own absorption is uniformly small relative to world sector size).

Assumption 4 is a normalization. Relaxing it costs nothing conceptually but requires carrying sector-size weights through every expression that follows, and we impose it for clarity, returning to the point in the conclusion. Assumption 5 formalizes the small open economy idealization implicit whenever a trade model treats one country's spending as negligible against the world market it trades in. The parameter ε indexes the quality of that idealization, and the bounds below are stated as explicit functions of ε rather than asserted only in the limit.

Proposition 3 (Export-Share Formula and Small-Country Limit). *Under Assumption 3, exactly,*

$$x_n^s = \frac{\omega_n^s (E^s - X_n^s)}{\sum_{s'} \omega_n^{s'} (E^{s'} - X_n^{s'})}.$$

If in addition Assumptions 4–5 hold, then

$$|x_n^s - \hat{x}_n^s| \leq \frac{\varepsilon}{1 - \varepsilon} \hat{x}_n^s,$$

so $x_n^s \rightarrow \hat{x}_n^s$ as $\varepsilon \rightarrow 0$.

The main result relating the two indices follows.

Theorem 1 (Nesting, with Explicit Remainder). *Under Assumptions 1, 3, 4, and 5,*

$$\sum_s x_n^s m_i^s = \text{SCI}_{ni} + R_{ni}, \quad |R_{ni}| \leq \frac{\varepsilon}{1 - \varepsilon} \text{SCI}_{ni}.$$

That is, in the limit $\varepsilon \rightarrow 0$, $\sum_s x_n^s m_i^s = \langle x_n, m_i \rangle \rightarrow \text{SCI}_{ni}$: the realized-composition inner product converges exactly to SCI_{ni} , and both are functionals of vectors that TCI itself compares.

The Michaely index and the inner product $\langle x_n, m_i \rangle$ are therefore two functionals of one and the same pair of vectors, a sum of minima in the first case and a sum of products in the second. The two are not interchangeable in general. When their arguments are shares confined to $[0, 1]$, however, one dominates the other, and the size of the gap depends on how far each argument sits from a corner of that interval.

Lemma 4 (The Minimum Dominates the Product). *For any $a, b \in [0, 1]$, $\min(a, b) \geq ab$, with equality if and only if $a \in \{0, 1\}$ or $b \in \{0, 1\}$. Consequently, for probability vectors $x_n, m_i \in [0, 1]^S$,*

$$\text{TCI}_{ni} = \sum_s \min(x_n^s, m_i^s) \geq \sum_s x_n^s m_i^s = \langle x_n, m_i \rangle,$$

with equality if and only if, for every sector s , $x_n^s \in \{0, 1\}$ or $m_i^s \in \{0, 1\}$.

Corollary 1 (Bound). *Under the hypotheses of Theorem 1, $\text{SCI}_{ni} \leq \text{TCI}_{ni} + |R_{ni}|$, and in the small-exporter limit $\varepsilon \rightarrow 0$, $\text{SCI}_{ni} \leq \text{TCI}_{ni}$ exactly, with equality if and only if, sector by sector, the exporter or the importer is completely specialized.*

Corollary 2 (Deterministic-Ricardian Limit). *It is a standard result ([Eaton and Kortum, 2002](#), extreme-value limit of the Fréchet distribution) that as $\theta \rightarrow \infty$ with Assumption 3 maintained and no exact cost ties, $\omega_n^s \rightarrow \{0, 1\}$ for each (n, s) : trade shares degenerate to deterministic Ricardian assignment. Then $x_n^s, m_i^s \rightarrow \{0, 1\}$ as well, the equality condition of Lemma 4 holds sector by sector, and $\text{SCI}_{ni} \rightarrow \text{TCI}_{ni}$ exactly. Complete specialization is precisely where the structural and empirical complementarity measures coincide.*

Remark 1. Corollary 2 identifies the sense in which the Michaely index approximates SCI_{ni} . The approximation is exact at the complete-specialization limit and nowhere else, and Corollary 1 measures the distance between the two elsewhere. Taken with Proposition 2, the three magnitudes stand in a fully specified relation to one another. Realized trade satisfies $G_{ni} = \Omega_n \text{SCI}_{ni}$ exactly under frictionless costs at any country size, while $\text{SCI}_{ni} \leq \text{TCI}_{ni}$ holds up to a remainder that vanishes as the exporter becomes small.

6. Structural Gravity with Complementarity

Proposition 2 relates SCI_{ni} to realized trade only at the frictionless benchmark. An empirical specification must accommodate trade costs that differ across country pairs. This section shows that, under two restrictions on how trade costs vary across sectors, the model yields a structural gravity equation in which SCI_{ni} enters as a regressor. That equation provides a baseline empirical specification for estimating the effect of complementarity on bilateral trade.

6.1 Assumptions

Importer i 's access to the world's sector- s suppliers is

$$a_i^s \equiv \sum_m \omega_m^s (d_{mi}^s)^{-\theta}, \quad (7)$$

an average of i 's trade costs with every supplier, weighted by the suppliers' world competitiveness in sector s .

Assumption 6 (Sector-Invariant Bilateral Costs). $d_{ni}^s = d_{ni}$ for every sector s , with $d_{nn} = 1$.

Assumption 7 (Sector-Invariant Supplier Access). $a_i^s = a_i$ for every sector s .

Assumption 6 allows trade costs to vary freely across country pairs but not across sectors. It replaces the uniform-cost structure of Assumption 2, under which costs are identical across pairs. Assumption 7 requires that an importer's access to suppliers, weighted by their competitiveness, be the same in every sector. It holds exactly at the frictionless benchmark and approximately whenever the competitive suppliers of each sector are, on average, equally costly to reach.

6.2 The Structural Gravity Equation

Proposition 4 (Structural Gravity with Complementarity). *Under Assumptions 6 and 7, for every pair $n \neq i$,*

$$X_{ni} = \frac{X_i}{a_i} \Omega_n d_{ni}^{-\theta} \text{SCI}_{ni}.$$

(Proof in Appendix B.)

Bilateral trade is the product of an importer term, an exporter term, a bilateral trade-cost term, and complementarity. At the frictionless benchmark, $d_{ni} = 1$ and $a_i = \sum_m \omega_m^s = 1$, so Proposition 4 reduces to Proposition 2.

6.3 The Baseline Empirical Specification

Taking logs of Proposition 4 and adding a year subscript,

$$\ln X_{nit} = \underbrace{\ln \Omega_{nt}}_{\text{exporter-year}} + \underbrace{\ln X_{it} - \ln a_{it}}_{\text{importer-year}} - \theta \ln d_{nit} + \ln \text{SCI}_{nit}.$$

The exporter term is absorbed by exporter-year fixed effects α_{nt} , the importer term by importer-year fixed effects γ_{it} , and $-\theta \ln d_{nit}$ is parameterized by bilateral trade-cost variables z_{nit} . Proposition 4 therefore yields the baseline empirical specification

$$X_{nit} = \exp(\alpha_{nt} + \gamma_{it} + \beta \ln \text{SCI}_{nit} + z'_{nit} \delta) \varepsilon_{nit}, \quad (8)$$

which can be estimated by PPML on all ordered pairs, zero flows included ([Santos Silva and Tenreyro, 2006](#)).

7. Identification: SCI as a Shift-Share Regressor

Definition 2 makes the shift-share structure of SCI_{ni} explicit: writing $\hat{x}_n^s$ for exporter n 's structural sectoral “shares” and s_i^s for importer i 's sectoral demand “shocks,”

$$\text{SCI}_{ni} = \sum_s \underbrace{\hat{x}_n^s}_{\text{exporter share}} \times \underbrace{s_i^s}_{\text{importer shock}}.$$

Each component is a function of one country's technology or demand fundamentals alone. Neither depends on trade within the pair, and neither can be computed from X_{ni} even in part. The claim is stronger than an assertion that SCI_{ni} happens to be uncorrelated with unobserved shocks to the (n, i) relationship, because no bilateral object of any kind enters the definition, leaving nothing through which such shocks could operate. Identification of the effect of SCI_{ni} on bilateral trade can accordingly appeal to the shift-share literature (Borusyak, Hull and Jaravel, 2022; Adão, Kolesár and Morales, 2019), with exogeneity argued from the many-sector exposure structure rather than from an excluded instrument. Section 8 sets out how $\hat{x}_n^s$ and s_i^s are each constructed from data on a single country.

8. Data and Construction of SCI

Constructing $\text{SCI}_{ni} = \sum_s \hat{x}_n^s s_i^s$ requires two objects, each built without reference to the bilateral (n, i) pair.

Exporter Side: $\hat{x}_n^s$. $T_n^s w_n^{-\theta}$ is not observed directly, but it is recoverable, up to normalization, from a sector-level structural gravity regression run separately by sector and year,

$$X_{ni}^s = \exp(\delta_n^s + \gamma_i^s - \theta \ln \bar{d}_{ni}^s) \cdot \eta_{ni}^s,$$

estimated by pseudo-Poisson maximum likelihood (Santos Silva and Tenreyro, 2006) because sectoral trade flows include zeros, where $\bar{d}_{ni}^s$ is built from observable gravity covariates (distance, contiguity, common language, colonial ties, and RTA membership from the CEPII Gravity database, together with tariffs from WITS) and η_{ni}^s is the residual. The exporter-sector fixed effect $\hat{\delta}_n^s$ is estimated by pooling over *all* of n 's trading partners in that sector and year, so it summarizes n 's competitiveness on average across many destinations, not its relationship with any one of them. Since $\hat{\delta}_n^s$ estimates $\ln(T_n^s w_n^{-\theta})$ up to a constant, exponentiating and normalizing across all exporters recovers the world market share,

$$\hat{\omega}_n^s = \frac{\exp(\hat{\delta}_n^s)}{\sum_m \exp(\hat{\delta}_m^s)}, \quad \hat{x}_n^s = \frac{\hat{\omega}_n^s}{\sum_{s'} \hat{\omega}_n^{s'}}.$$

The trade elasticity θ need not be pinned down separately at this stage, since it is absorbed into $\hat{\delta}_n^s$ and cancels in the ratio defining $\hat{\omega}_n^s$. It becomes necessary only for welfare or counterfactual exercises outside the scope of this paper, for which we would follow Caliendo and Parro (2015). One refinement remains. Pooling over n 's partners leaves a residual dependence of order $1/N$ on the

pair itself, since the observation for (n, i) is among those used to estimate $\hat{\delta}_n^s$. Constructing SCI_{ni} for a given importer therefore calls for a leave-one-out fixed effect, estimated after dropping that importer's own observation, following the leave-one-out share construction that [Borusyak, Hull and Jaravel \(2022\)](#) recommend for shift-share designs generally.

Importer Side: s_i^s . Lemma 1 gives $s_i^s = \beta_i^s$, a demand primitive unrelated to n and to trade flows generally. The appropriate data are therefore sectoral final-demand or absorption figures from national accounts and input-output tables, such as the OECD Inter-Country Input-Output tables, the World Input-Output Database, or GTAP, all of which report a country's consumption of each sector's output by source without reference to any particular partner. Constructed this way, s_i^s requires no assumption linking it to observed trade, because it never passes through a trade identity. Where input-output coverage is missing for a country-year, Assumption 1 licenses a fallback in which m_i^s , computed from BACI and excluding exporter n from the sum that defines it, stands in for s_i^s exactly.

Construction. With $\hat{x}_n^s$ and s_i^s in hand, $\hat{\text{SCI}}_{ni} = \sum_s \hat{x}_n^s s_i^s$ follows immediately, and neither term draws on trade between n and i . Realized trade $G_{ni} = X_{ni}/X_i$, or X_{ni} itself, is observed directly from BACI, and Proposition 2 ties it to SCI_{ni} when cross-border costs vanish, and Proposition 4 when trade costs vary across country pairs.

9. A Numerical Illustration

This section works through a fully specified example, small enough to verify by hand, in which the bias formula of Proposition 1, the export-share formula of Proposition 3, the relationship to realized trade in Proposition 2, the decomposition in Theorem 1, and the inequality of Lemma 4 are each checked against direct computation from primitives. Every value reported below follows from the definitions in Sections 2 through 5.

9.1 The Bias Formula Away from the Frictionless Benchmark

Take three countries A, B, C and a single sector with $\theta = 2$, wages normalized to $w_A = w_B = w_C = 1$, and technology $T_A = 4, T_B = 1, T_C = 1$, so $\Phi = T_A + T_B + T_C = 6$ and the world frictionless shares are $\omega_A = 4/6 = 0.6667$, $\omega_B = \omega_C = 1/6 = 0.1667$. Impose Assumption 2 with a common cross-border cost $\delta = 2$, so $\kappa = \delta^{-\theta} = 2^{-2} = 0.25$.

Direct computation from (1), with $d_{BB} = 1$ and $d_{AB} = d_{CB} = \delta = 2$, gives

$$\Phi_B = T_B(w_B \cdot 1)^{-2} + T_A(w_A \cdot 2)^{-2} + T_C(w_C \cdot 2)^{-2} = 1 + 4(0.25) + 1(0.25) = 2.25,$$

$$\pi_{AB} = \frac{T_A(w_A \cdot 2)^{-2}}{\Phi_B} = \frac{4 \times 0.25}{2.25} = \frac{1}{2.25} = 0.4444.$$

Proposition 1's closed form gives, independently,

$$\pi_{AB} = \frac{\kappa \omega_A}{\kappa + (1 - \kappa)\omega_B} = \frac{0.25 \times 0.6667}{0.25 + 0.75 \times 0.1667} = \frac{0.1667}{0.375} = 0.4444,$$

matching the direct computation exactly. The bias is $\pi_{AB} - \omega_A = 0.4444 - 0.6667 = -0.2222$, and the proposition's bound gives $\omega_A \omega_B (1 - \kappa) / \kappa = 0.6667 \times 0.1667 \times 3 = 0.3333$, which indeed bounds the realized bias in absolute value ($0.2222 \leq 0.3333$).

9.2 The Nesting Theorem at the Frictionless Benchmark

Now take three countries A, B, C and two sectors under Assumption 3 ($\kappa^s = 1$), with technology $T_A^1 = 4, T_A^2 = 1$; $T_B^1 = 1, T_B^2 = 4$; $T_C^1 = 1, T_C^2 = 2$ (wages again normalized to 1), so $\Phi^1 = 6, \Phi^2 = 7$ and

$$\omega_A = (0.6667, 0.1429), \quad \omega_B = (0.1667, 0.5714), \quad \omega_C = (0.1667, 0.2857).$$

Absorption is $X_A = (30, 45)$, $X_B = (45, 30)$, $X_C = (15, 15)$, giving equal sector sizes $E^1 = E^2 = 90$ (Assumption 4) and total spending $X_A = X_B = 75$, $X_C = 30$: C is the small country in this example ($\varepsilon_C = \max_s X_C^s / E^s = 1/6$), while A and B are not ($\varepsilon_A = \varepsilon_B = 1/2$). Assumption 1 is imposed directly by setting $m_i^s = s_i^s$ for every country, so $s_A = (0.4, 0.6)$, $s_B = (0.6, 0.4)$.

Proposition 3's exact formula gives realized export shares

$$x_A = \left(\frac{56}{65}, \frac{9}{65} \right) = (0.8615, 0.1385),$$

$$x_B = \left(\frac{7}{39}, \frac{32}{39} \right) = (0.1795, 0.8205),$$

$$x_C = \left(\frac{7}{19}, \frac{12}{19} \right) = (0.3684, 0.6316),$$

each obtained by substituting ω_n^s , E^s , and X_n^s into the formula rather than by approximation. Table 1 compares these with the structural composition $\hat{x}_n^s = \omega_n^s / \Omega_n$ and displays the pattern the theorem predicts. For C the two agree exactly, since that country's absorption is split evenly across sectors, $X_C^1 = X_C^2$, which by the proof of Proposition 3 sets the deviation term to zero whatever the value of ε_C . For A and B they differ by several percentage points, these being the countries whose absorption is both larger and more skewed across sectors, the two features that Assumption 5 restricts.

Table 1: Realized export shares versus the structural composition

Exporter n	x_n^1 (realized)	$\hat{x}_n^1$ (structural)	absolute deviation
A	0.8615	0.8235	0.0380
B	0.1795	0.2258	0.0463
C	0.3684	0.3684	0.0000

Now take the bilateral pair $n = A$ (exporter), $i = B$ (importer), with $\Omega_A = \omega_A^1 + \omega_A^2 = 0.8095$ and $\hat{x}_A = (\omega_A^1 / \Omega_A, \omega_A^2 / \Omega_A) = (0.8235, 0.1765)$. By Definition 2,

$$\text{SCI}_{AB} = \hat{x}_A^1 s_B^1 + \hat{x}_A^2 s_B^2 = 0.8235 \times 0.6 + 0.1765 \times 0.4 = 0.4941 + 0.0706 = 0.5647.$$

Proposition 2 requires $G_{AB} = \Omega_A \text{SCI}_{AB}$ to hold exactly. Under Assumption 3 we have $\pi_{AB}^s = \omega_A^s$, so

that $G_{AB} = \omega_A^1 s_B^1 + \omega_A^2 s_B^2 = 0.6667 \times 0.6 + 0.1429 \times 0.4 = 0.4571$, while $\Omega_A \text{SCI}_{AB} = 0.8095 \times 0.5647 = 0.4571$. The two agree with no residual.

Turning to the Michaely index,

$$\text{TCI}_{AB} = \sum_s \min(x_A^s, m_B^s) = \min(0.8615, 0.6) + \min(0.1385, 0.4) = 0.6 + 0.1385 = 0.7385,$$

while the inner product of the realized compositions is $\langle x_A, m_B \rangle = 0.8615 \times 0.6 + 0.1385 \times 0.4 = 0.5169 + 0.0554 = 0.5723$. Lemma 4 requires $\text{TCI}_{AB} \geq \langle x_A, m_B \rangle$, and indeed $0.7385 \geq 0.5723$. Theorem 1 in turn requires the inner product to equal SCI_{AB} up to a bounded residual,

$$R_{AB} = \langle x_A, m_B \rangle - \text{SCI}_{AB} = 0.5723 - 0.5647 = 0.0076,$$

well inside the theorem's bound of $\frac{\varepsilon_A}{1-\varepsilon_A} \text{SCI}_{AB} = 1 \times 0.5647 = 0.5647$. The bound is loose here because A is not a small exporter, with $\varepsilon_A = 0.5$, and it tightens as ε falls, as the row for C in Table 1 already shows for the export share. Comparing the indices directly gives $\text{SCI}_{AB} = 0.5647 \leq \text{TCI}_{AB} = 0.7385$, which confirms Corollary 1 numerically even away from the strict small-exporter limit and without any scaling factor to make the two comparable.

The equality condition in Lemma 4 can be checked on degenerate vectors. Suppose n exported only in sector 1, so that $x_n = (1, 0)$, and i imported only in sector 1, so that $m_i = (1, 0)$. Then $\text{TCI}_{ni} = \min(1, 1) + \min(0, 0) = 1 = 1 \times 1 + 0 \times 0 = \langle x_n, m_i \rangle$, which reproduces the complete-specialization limit of Corollary 2.

10. Extension: Non-Homothetic Demand

Every result from Section 3 onward rests on Lemma 1, which permits s_i^s to be treated as a trade-cost-independent primitive alongside the exporter's technology parameters. The lemma follows from the Cobb–Douglas upper tier assumed in Section 2 and fails under richer demand systems. With non-homothetic CES preferences (Fieler, 2011), expenditure shares become $s_i^s(Y_i)$ and vary with income and relative prices, so that rich and poor importers demand different sectoral mixes even when facing identical prices, and observed shares cease to stand in for the underlying taste parameters.

Such an extension would make SCI_{ni} income-dependent and would give complementarity a development dimension, which is suggestive given that intra-African pairs, disproportionately low-income, record the lowest measured values of the Michaely index. It would also forfeit the measurement convenience of Lemma 1, since recovering demand primitives would then require estimation exploiting income or price variation across importers or over time. We leave the extension to future work. It alters the demand side of Section 2 while leaving the competitiveness side of Sections 4.2 through 5 untouched, and Theorem 1 requires only Assumption 1 rather than homotheticity on the demand side.

11. Conclusion

A multi-sector Eaton–Kortum economy yields a structural measure of trade complementarity: the alignment between an exporter’s technology-driven comparative-advantage profile and an importer’s demand composition, each constructed from a single country’s own characteristics. The measure was not designed to resemble the Michaely index already in wide use, nor is it a repaired version of a bilateral trade share. Its two ingredients are chosen so that neither can be recovered from trade between the exporter and importer being scored. Because the two indices apply different functionals to different arguments, no assumption renders them identical.

The paper instead establishes a precise sense in which they coincide, together with a separate and exact relation between the structural measure and trade actually observed. One restriction on the import side and two on the export side connect SCI to an overlap of the same pair of vectors the Michaely index compares, up to a remainder bounded explicitly and vanishing in the small-exporter limit. The sum-of-minima functional weakly dominates that overlap, with equality holding exactly when the exporter or importer is completely specialized sector by sector, which is the deterministic Ricardian limit of the underlying technology. Under frictionless costs and at any country size, realized bilateral trade equals SCI scaled by the exporter’s aggregate world competitiveness, with no error term at all. Complementarity and realized trade are thus distinct magnitudes joined by a single factor once geography is set aside. Under two restrictions on how trade costs vary across sectors, the model yields a structural gravity equation that provides a baseline empirical specification for estimating the effect of complementarity on bilateral trade.

The claim advanced here is deliberately bounded. SCI is a theory-consistent measure built from predetermined fundamentals rather than realized bilateral trade, which removes the circularity a naive bilateral construction would carry and supports the shift-share identification argument of Section 7. It rationalizes and bounds the descriptive index rather than reproducing it, and away from the small-exporter and complete-specialization benchmarks the two measures diverge. Three limitations deserve emphasis. The assumption of equal world sector sizes is a normalization rather than an economic primitive. The non-homothetic extension of Section 10 is not proved and would require demand estimation in place of a direct reading of absorption data. The leave-one-out construction of Section 8 remains to be implemented on data. With the index established on structural foundations, its causal identification and its use in valuing market access are natural next steps.

Appendix A: Derivation of the Trade-Share Equation

This appendix derives the trade-share equation (1) from the underlying Fréchet efficiency draws, following [Eaton and Kortum \(2002\)](#).

Setup. Fix a sector s and an importer i ; all quantities below implicitly carry the superscript s . Normalize the continuum of goods in the sector to $j \in [0, 1]$. As stated in Section 2, every country m can produce every good j , and its efficiency $z_m(j)$ is drawn independently across m and j from a Fréchet distribution with shape parameter $\theta > 1$ and country-sector-specific scale parameter T_m^s :

$$\Pr[z_m(j) \leq z] = \exp(-T_m^s z^{-\theta}), \quad z > 0.$$

Producing one unit of good j in country m costs $w_m/z_m(j)$ (the wage divided by efficiency), and shipping it to i multiplies this by the iceberg cost d_{mi}^s , so the price m could deliver good j to i at is

$$c_m^i(j) = \frac{w_m d_{mi}^s}{z_m(j)}.$$

Step 1: the distribution of the delivered price $c_m^i(j)$. For any $c > 0$,

$$\Pr[c_m^i(j) \leq c] = \Pr\left[z_m(j) \geq \frac{w_m d_{mi}^s}{c}\right] = 1 - \Pr\left[z_m(j) < \frac{w_m d_{mi}^s}{c}\right] = 1 - \exp\left(-T_m^s \left(\frac{w_m d_{mi}^s}{c}\right)^{-\theta}\right),$$

using the Fréchet CDF from the Setup with $z = w_m d_{mi}^s / c$. Simplifying the exponent,

$$T_m^s \left(\frac{w_m d_{mi}^s}{c}\right)^{-\theta} = T_m^s (w_m d_{mi}^s)^{-\theta} c^\theta,$$

so, writing $\phi_m^i \equiv T_m^s (w_m d_{mi}^s)^{-\theta}$,

$$\Pr[c_m^i(j) \leq c] = 1 - \exp(-\phi_m^i c^\theta). \quad (9)$$

Step 2: the price actually paid by i . Under perfect competition across the N potential suppliers, i buys good j from whichever country offers it cheapest, so the price paid is $p_i(j) = \min_{m=1, \dots, N} c_m^i(j)$. Since the efficiency draws $z_m(j)$ are independent across m , so are the prices $c_m^i(j)$, and

$$\Pr[p_i(j) > p] = \Pr\left[\bigcap_{m=1}^N \{c_m^i(j) > p\}\right] = \prod_{m=1}^N \Pr[c_m^i(j) > p] = \prod_{m=1}^N \exp(-\phi_m^i p^\theta),$$

using (9). Summing the exponents,

$$\Pr[p_i(j) > p] = \exp\left(-p^\theta \sum_{m=1}^N \phi_m^i\right) = \exp(-\Phi_i^s p^\theta), \quad \Phi_i^s \equiv \sum_{m=1}^N \phi_m^i = \sum_{m=1}^N T_m^s (w_m d_{mi}^s)^{-\theta}, \quad (10)$$

which is exactly the denominator Φ_i^s of (1).

Step 3: the probability that n is the least-cost supplier. By (9), the density of $c_n^i(j)$ at p is the derivative of $1 - \exp(-\phi_n^i p^\theta)$ with respect to p , namely $\theta \phi_n^i p^{\theta-1} \exp(-\phi_n^i p^\theta)$. Country n is the least-cost supplier of good j exactly when $c_n^i(j) = p$ for some p and every other country's price exceeds p ; integrating over p ,

$$\pi_{ni}^s = \Pr[c_n^i(j) < c_m^i(j) \ \forall m \neq n] = \int_0^\infty \theta \phi_n^i p^{\theta-1} \exp(-\phi_n^i p^\theta) \prod_{m \neq n} \Pr[c_m^i(j) > p] dp.$$

By (9), $\prod_{m \neq n} \Pr[c_m^i(j) > p] = \exp(-(\Phi_i^s - \phi_n^i) p^\theta)$ (the same product as in Step 2, with the $m = n$ factor removed), so the integrand's two exponentials combine:

$$\pi_{ni}^s = \int_0^\infty \theta \phi_n^i p^{\theta-1} \exp(-\phi_n^i p^\theta) \exp(-(\Phi_i^s - \phi_n^i) p^\theta) dp = \int_0^\infty \theta \phi_n^i p^{\theta-1} \exp(-\Phi_i^s p^\theta) dp.$$

Substitute $u = \Phi_i^s p^\theta$, so $du = \theta \Phi_i^s p^{\theta-1} dp$, i.e. $\theta p^{\theta-1} dp = du/\Phi_i^s$; as p ranges over $(0, \infty)$, so does u . Substituting,

$$\pi_{ni}^s = \frac{\phi_n^i}{\Phi_i^s} \int_0^\infty e^{-u} du = \frac{\phi_n^i}{\Phi_i^s},$$

since $\int_0^\infty e^{-u} du = 1$. Substituting back $\phi_n^i = T_n^s(w_n d_{ni}^s)^{-\theta}$ gives

$$\pi_{ni}^s = \frac{T_n^s(w_n d_{ni}^s)^{-\theta}}{\Phi_i^s}, \quad (11)$$

which is exactly (1).

Step 4: from a sourcing probability to an expenditure share. Steps 1–3 show that π_{ni}^s is the *probability*, for a single good j , that n is the cheapest source. With a continuum of goods $j \in [0, 1]$ drawn independently, the law of large numbers implies that the *fraction* of goods actually sourced from n equals π_{ni}^s almost surely. Under the Cobb–Douglas demand aggregator of Section 2, in which every good within a sector receives the same expenditure weight ex ante regardless of its realized source, the expenditure share on goods sourced from n equals that fraction, and hence equals π_{ni}^s ; see [Eaton and Kortum \(2002\)](#) for the demand-side argument in full. This identifies π_{ni}^s in (11) with i 's expenditure share on n in sector s , completing the derivation of (1). Finally, $\sum_{n=1}^N \pi_{ni}^s = \sum_{n=1}^N T_n^s(w_n d_{ni}^s)^{-\theta} / \Phi_i^s = \Phi_i^s / \Phi_i^s = 1$ by the definition of Φ_i^s in (10), confirming that $\{\pi_{ni}^s\}_{n=1}^N$ is a valid vector of expenditure shares.

Appendix B: Proofs of Main Results

This appendix collects the proofs of the results stated in the main text, in order of appearance.

Proof of Lemma 1. Let P_i^s denote the price index for sector s 's composite good in country i . Country i maximizes $\sum_s \beta_i^s \ln C_i^s$ subject to $\sum_s P_i^s C_i^s = Y_i$. The Lagrangian is $\mathcal{L} = \sum_s \beta_i^s \ln C_i^s + \lambda(Y_i - \sum_s P_i^s C_i^s)$, with first-order condition $\beta_i^s / C_i^s = \lambda P_i^s$ for every s , i.e. $C_i^s = \beta_i^s / (\lambda P_i^s)$. Substituting into the budget constraint, $\sum_s P_i^s \cdot \beta_i^s / (\lambda P_i^s) = Y_i$, so $\sum_s \beta_i^s / \lambda = Y_i$; since $\sum_s \beta_i^s = 1$, this gives

$\lambda = 1/Y_i$. Hence $C_i^s = \beta_i^s Y_i / P_i^s$, and expenditure on sector s is $X_i^s = P_i^s C_i^s = P_i^s \cdot \beta_i^s Y_i / P_i^s = \beta_i^s Y_i$: the price index P_i^s cancels exactly, for any value it takes. Dividing by $X_i = \sum_s X_i^s = Y_i \sum_s \beta_i^s = Y_i$ (using $\sum_s \beta_i^s = 1$) gives $s_i^s = X_i^s / X_i = \beta_i^s Y_i / Y_i = \beta_i^s$. $\square$

Proof of Lemma 2. *Step 1: a pointwise identity.* Fix s and let $a = a_s, b = b_s \geq 0$. There are two exhaustive cases.

Case $a \geq b$. Then $\min(a, b) = b$ and $|a - b| = a - b$ (since $a - b \geq 0$). Compute $a + b - 2\min(a, b) = a + b - 2b = a - b = |a - b|$.

Case $a < b$. Then $\min(a, b) = a$ and $|a - b| = b - a$ (since $a - b < 0$, so $|a - b| = -(a - b) = b - a$). Compute $a + b - 2\min(a, b) = a + b - 2a = b - a = |a - b|$.

Both cases give the same identity, so for every $a, b \geq 0$,

$$|a - b| = a + b - 2\min(a, b). \quad (12)$$

Step 2: sum over sectors. Apply (12) with $a = a_s, b = b_s$ for each $s = 1, \dots, S$ and sum both sides over s :

$$\sum_{s=1}^S |a_s - b_s| = \sum_{s=1}^S a_s + \sum_{s=1}^S b_s - 2 \sum_{s=1}^S \min(a_s, b_s).$$

By hypothesis $\sum_s a_s = 1$ and $\sum_s b_s = 1$, so the right-hand side equals $1 + 1 - 2 \sum_s \min(a_s, b_s) = 2 - 2 \sum_s \min(a_s, b_s)$. Hence

$$\sum_{s=1}^S |a_s - b_s| = 2 - 2 \sum_{s=1}^S \min(a_s, b_s). \quad (13)$$

Step 3: solve for the sum of minima. Divide (13) by 2 and rearrange: $\frac{1}{2} \sum_s |a_s - b_s| = 1 - \sum_s \min(a_s, b_s)$, so

$$1 - \frac{1}{2} \sum_{s=1}^S |a_s - b_s| = 1 - \left[1 - \sum_{s=1}^S \min(a_s, b_s) \right] = \sum_{s=1}^S \min(a_s, b_s).$$

This proves the general identity for any two vectors with entries summing to 1.

Step 4: apply to TCI. By construction (Section 3), $x_n^s \geq 0$ for every s and $\sum_s x_n^s = 1$ (it is $X_n^{\text{exp},s}$ divided by its own total across s), and likewise $m_i^s \geq 0$ with $\sum_s m_i^s = 1$. Setting $a_s = x_n^s$ and $b_s = m_i^s$ in the identity just proved, and recalling Definition 1, $\text{TCI}_{ni} = 1 - \frac{1}{2} \sum_s |x_n^s - m_i^s|$, gives $\text{TCI}_{ni} = \sum_s \min(x_n^s, m_i^s)$, as claimed. $\square$

Proof of Lemma 3. *Step 1: express imports as absorption net of the domestic term.* By the definition of $X_i^{\text{imp},s}$ in Section 3, i 's sector- s imports are total absorption minus what i buys from itself: $X_i^{\text{imp},s} = X_i^s - X_{ii}^s$. By the trade-share equation, $X_{ii}^s = \pi_{ii}^s X_i^s$ (this is $X_{ni}^s = \pi_{ni}^s X_i^s$ evaluated at $n = i$). Substituting,

$$X_i^{\text{imp},s} = X_i^s - \pi_{ii}^s X_i^s = X_i^s (1 - \pi_{ii}^s).$$

Step 2: impose Assumption 1. The assumption states $\pi_{ii}^s = \pi_{ii}$ for every s , i.e. the same number $\pi_{ii} \in [0, 1)$ regardless of sector. Substituting this constant in place of π_{ii}^s in Step 1,

$$X_i^{\text{imp},s} = (1 - \pi_{ii}) X_i^s \quad \text{for every } s. \quad (14)$$

Step 3: sum over sectors to get total imports. Summing (14) over $s = 1, \dots, S$, and using that $(1 - \pi_{ii})$ does not depend on s so it factors out of the sum,

$$X_i^{\text{imp}} \equiv \sum_s X_i^{\text{imp},s} = (1 - \pi_{ii}) \sum_s X_i^s = (1 - \pi_{ii}) X_i,$$

using the definition $X_i \equiv \sum_s X_i^s$ from Section 2.

Step 4: form the ratio. By definition $m_i^s = X_i^{\text{imp},s} / X_i^{\text{imp}}$. Substituting the results of Steps 2 and 3,

$$m_i^s = \frac{(1 - \pi_{ii}) X_i^s}{(1 - \pi_{ii}) X_i}.$$

Because $\pi_{ii} < 1$ whenever country i sources a strictly positive share of at least one sector from abroad, the factor $(1 - \pi_{ii})$ is a nonzero number common to numerator and denominator and cancels:

$$m_i^s = \frac{X_i^s}{X_i} = s_i^s,$$

the last equality by the definition $s_i^s \equiv X_i^s / X_i$ from Section 2. This holds for every s , which is the claim. $\square$

Proof of Proposition 1. Fix a sector s and an importer i ; all quantities below implicitly carry the superscript s except where written out. Recall from Section 2 that $T_m^s > 0$ and $w_m > 0$ for every m , so $\Phi^s = \sum_m T_m^s w_m^{-\theta} > 0$ and $\Phi_i^s > 0$: every division below is by a strictly positive number.

Step 1: split Φ_i^s into the domestic and cross-border pieces. By definition, the sum $\Phi_i^s = \sum_{m=1}^N T_m^s (w_m d_{mi}^s)^{-\theta}$ runs over *all* countries m , including $m = i$. Separate the $m = i$ term from the rest of the sum:

$$\Phi_i^s = T_i^s (w_i d_{ii}^s)^{-\theta} + \sum_{m \neq i} T_m^s (w_m d_{mi}^s)^{-\theta}.$$

By the domestic-cost convention $d_{ii}^s = 1$ (Section 2), the first term is $T_i^s (w_i \cdot 1)^{-\theta} = T_i^s w_i^{-\theta}$. By Assumption 2, $d_{mi}^s = \delta^s$ for every $m \neq i$ (a single number not depending on m), so every term in the remaining sum has the same cost factor $(\delta^s)^{-\theta}$, which can be factored out:

$$\Phi_i^s = T_i^s w_i^{-\theta} + (\delta^s)^{-\theta} \sum_{m \neq i} T_m^s w_m^{-\theta}.$$

Step 2: rewrite in terms of κ^s and Φ^s . By definition $\kappa^s \equiv (\delta^s)^{-\theta}$. Also, $\sum_{m \neq i} T_m^s w_m^{-\theta}$ is the full sum $\Phi^s = \sum_m T_m^s w_m^{-\theta}$ with the $m = i$ term $T_i^s w_i^{-\theta}$ removed, i.e. $\sum_{m \neq i} T_m^s w_m^{-\theta} = \Phi^s - T_i^s w_i^{-\theta}$.

Substituting both into Step 1,

$$\Phi_i^s = T_i^s w_i^{-\theta} + \kappa^s (\Phi^s - T_i^s w_i^{-\theta}).$$

Expand the product and collect the two terms containing $T_i^s w_i^{-\theta}$:

$$\Phi_i^s = T_i^s w_i^{-\theta} + \kappa^s \Phi^s - \kappa^s T_i^s w_i^{-\theta} = \kappa^s \Phi^s + (1 - \kappa^s) T_i^s w_i^{-\theta}.$$

By the definition $\omega_i^s \equiv T_i^s w_i^{-\theta} / \Phi^s$, we have $T_i^s w_i^{-\theta} = \omega_i^s \Phi^s$. Substituting this and factoring out Φ^s from both remaining terms,

$$\Phi_i^s = \kappa^s \Phi^s + (1 - \kappa^s) \omega_i^s \Phi^s = \Phi^s [\kappa^s + (1 - \kappa^s) \omega_i^s]. \quad (15)$$

Step 3: compute π_{ni}^s for $n \neq i$. By (1), $\pi_{ni}^s = T_n^s (w_n d_{ni}^s)^{-\theta} / \Phi_i^s$. Since $n \neq i$, Assumption 2 gives $d_{ni}^s = \delta^s$, so the numerator is $T_n^s w_n^{-\theta} (\delta^s)^{-\theta} = \kappa^s T_n^s w_n^{-\theta}$. Using $T_n^s w_n^{-\theta} = \omega_n^s \Phi^s$ (same definition as above, applied to n) and substituting the denominator from (15),

$$\pi_{ni}^s = \frac{\kappa^s \omega_n^s \Phi^s}{\Phi^s [\kappa^s + (1 - \kappa^s) \omega_i^s]}.$$

The factor Φ^s appears in both numerator and denominator and cancels (it is strictly positive, so this is a valid cancellation), leaving

$$\pi_{ni}^s = \frac{\kappa^s \omega_n^s}{\kappa^s + (1 - \kappa^s) \omega_i^s}, \quad (16)$$

which is the first claim of the proposition.

Step 4: compute the deviation from ω_n^s . Subtract ω_n^s from both sides of (16) and factor out ω_n^s :

$$\pi_{ni}^s - \omega_n^s = \omega_n^s \left[\frac{\kappa^s}{\kappa^s + (1 - \kappa^s) \omega_i^s} - 1 \right].$$

Write the bracketed term over the common denominator $\kappa^s + (1 - \kappa^s) \omega_i^s$:

$$\frac{\kappa^s}{\kappa^s + (1 - \kappa^s) \omega_i^s} - 1 = \frac{\kappa^s - [\kappa^s + (1 - \kappa^s) \omega_i^s]}{\kappa^s + (1 - \kappa^s) \omega_i^s} = \frac{-(1 - \kappa^s) \omega_i^s}{\kappa^s + (1 - \kappa^s) \omega_i^s}.$$

Substituting back,

$$\pi_{ni}^s - \omega_n^s = -\omega_n^s \cdot \frac{(1 - \kappa^s) \omega_i^s}{\kappa^s + (1 - \kappa^s) \omega_i^s}. \quad (17)$$

Step 5: bound the deviation. By definition $\kappa^s = (\delta^s)^{-\theta} \in (0, 1]$ (since $\delta^s \geq 1$ and $\theta > 1$), so $1 - \kappa^s \geq 0$; and $\omega_i^s \in [0, 1]$ is a market share, so $\omega_i^s \geq 0$. Hence the denominator of (17) satisfies

$$\kappa^s + (1 - \kappa^s) \omega_i^s \geq \kappa^s + 0 = \kappa^s > 0,$$

because we are adding a nonnegative quantity $(1 - \kappa^s)\omega_i^s \geq 0$ to κ^s . Taking absolute values in (17) (all of $\omega_n^s, \omega_i^s, 1 - \kappa^s$ are nonnegative, so only the denominator's size matters) and using that a positive fraction only increases when its denominator shrinks,

$$|\pi_{ni}^s - \omega_n^s| = \omega_n^s \omega_i^s \frac{1 - \kappa^s}{\kappa^s + (1 - \kappa^s)\omega_i^s} \leq \omega_n^s \omega_i^s \frac{1 - \kappa^s}{\kappa^s},$$

which is the claimed bound.

Step 6: the frictionless case. If $\kappa^s = 1$ (equivalently $\delta^s = 1$), the right-hand side of the bound is $\omega_n^s \omega_i^s \cdot 0/1 = 0$, so $\pi_{ni}^s = \omega_n^s$ exactly for every $i \neq n$; equivalently, setting $\kappa^s = 1$ directly in (16) gives $\pi_{ni}^s = \omega_n^s/[1 + 0 \cdot \omega_i^s] = \omega_n^s$. $\square$

Proof of Proposition 2. By Definition 2, $\text{SCI}_{ni} = \sum_s \hat{x}_n^s s_i^s$. By Definition (4), $\hat{x}_n^s = \omega_n^s/\Omega_n$, so

$$\Omega_n \text{SCI}_{ni} = \Omega_n \sum_s \frac{\omega_n^s}{\Omega_n} s_i^s = \sum_s \omega_n^s s_i^s,$$

the Ω_n factors canceling termwise since Ω_n does not depend on s . By Assumption 3, Proposition 1 (Step 6 of its proof) gives $\pi_{ni}^s = \omega_n^s$ exactly for $n \neq i$, so substituting into equation (5),

$$G_{ni} = \sum_s \pi_{ni}^s s_i^s = \sum_s \omega_n^s s_i^s = \Omega_n \text{SCI}_{ni},$$

the middle equality using $\pi_{ni}^s = \omega_n^s$ termwise and the last equality by the display above. This holds for any X_n^s , i.e. at any country size, since no assumption beyond Assumption 3 was used. $\square$

Proof of Proposition 3. *Step 1: exact formula for $X_n^{\text{exp},s}$.* By Assumption 3, Assumption 2 holds with $\kappa^s = 1$ for every s ; Proposition 1 (Step 6 of its proof) then gives $\pi_{ni}^s = \omega_n^s$ exactly, for every $i \neq n$ and every s , a single number that does not depend on which importer $i \neq n$ we look at. By the definition $X_n^{\text{exp},s} = \sum_{i \neq n} X_{ni}^s$ (Section 3) and $X_{ni}^s = \pi_{ni}^s X_i^s$ (Section 2),

$$X_n^{\text{exp},s} = \sum_{i \neq n} \pi_{ni}^s X_i^s = \sum_{i \neq n} \omega_n^s X_i^s.$$

Since ω_n^s does not depend on i , it factors out of the sum over i :

$$X_n^{\text{exp},s} = \omega_n^s \sum_{i \neq n} X_i^s.$$

The sum $\sum_{i \neq n} X_i^s$ is the sum over *all* countries i , $E^s = \sum_i X_i^s$, with the $i = n$ term X_n^s removed, i.e. $\sum_{i \neq n} X_i^s = E^s - X_n^s$. Hence

$$X_n^{\text{exp},s} = \omega_n^s (E^s - X_n^s), \tag{18}$$

exactly, using only Assumption 3.

Step 2: the export-share formula. By definition $x_n^s = X_n^{\text{exp},s}/X_n^{\text{exp}}$ with $X_n^{\text{exp}} = \sum_{s'} X_n^{\text{exp},s'}$.

Substituting (18) in the numerator and in every term of the denominator's sum,

$$x_n^s = \frac{\omega_n^s(E^s - X_n^s)}{\sum_{s'} \omega_n^{s'}(E^{s'} - X_n^{s'})},$$

which is the exact (non-limiting) formula claimed.

Step 3: impose equal sector size. Now additionally impose Assumption 4, $E^s = E$ for every s . Substituting $E^s = E$ (and $E^{s'} = E$) into the Step 2 formula,

$$x_n^s = \frac{\omega_n^s(E - X_n^s)}{\sum_{s'} \omega_n^{s'}(E - X_n^{s'})}.$$

Define $D \equiv \sum_{s'} \omega_n^{s'}(E - X_n^{s'})$, the denominator above, so $x_n^s = \omega_n^s(E - X_n^s)/D$. Expand D by distributing the sum:

$$D = \sum_{s'} \omega_n^{s'} E - \sum_{s'} \omega_n^{s'} X_n^{s'} = E \sum_{s'} \omega_n^{s'} - \sum_{s'} \omega_n^{s'} X_n^{s'} = E\Omega_n - \sum_{s'} \omega_n^{s'} X_n^{s'},$$

using $\Omega_n \equiv \sum_{s'} \omega_n^{s'}$ and that E does not depend on s' so it factors out of the first sum.

Step 4: the deviation from $\hat{x}_n^s = \omega_n^s/\Omega_n$. We want to bound $x_n^s - \omega_n^s/\Omega_n$. Write both terms over the common denominator $D\Omega_n$:

$$x_n^s - \frac{\omega_n^s}{\Omega_n} = \frac{\omega_n^s(E - X_n^s)}{D} - \frac{\omega_n^s}{\Omega_n} = \frac{\omega_n^s(E - X_n^s)\Omega_n}{D\Omega_n} - \frac{\omega_n^s D}{D\Omega_n} = \frac{\omega_n^s}{D\Omega_n} [(E - X_n^s)\Omega_n - D].$$

Now expand the bracket, substituting the Step 3 expression for D :

$$(E - X_n^s)\Omega_n - D = E\Omega_n - X_n^s\Omega_n - \left(E\Omega_n - \sum_{s'} \omega_n^{s'} X_n^{s'}\right) = E\Omega_n - X_n^s\Omega_n - E\Omega_n + \sum_{s'} \omega_n^{s'} X_n^{s'}.$$

The two $E\Omega_n$ terms cancel. Writing $X_n^s\Omega_n = X_n^s \sum_{s'} \omega_n^{s'} = \sum_{s'} \omega_n^{s'} X_n^s$ (again using $\Omega_n = \sum_{s'} \omega_n^{s'}$, now moving X_n^s , which does not depend on the summation index s' , inside the sum), the remaining two sums combine term by term:

$$(E - X_n^s)\Omega_n - D = \sum_{s'} \omega_n^{s'} X_n^{s'} - \sum_{s'} \omega_n^{s'} X_n^s = \sum_{s'} \omega_n^{s'} (X_n^{s'} - X_n^s). \quad (19)$$

Step 5: bound using the small-exporter assumption. By Assumption 5, $0 \leq X_n^{s'} \leq \varepsilon E$ for every s' , and in particular also for s , so $0 \leq X_n^s \leq \varepsilon E$. Hence for every pair s, s' ,

$$-\varepsilon E \leq X_n^{s'} - X_n^s \leq \varepsilon E, \quad \text{i.e.} \quad |X_n^{s'} - X_n^s| \leq \varepsilon E.$$

Applying the triangle inequality to (19) and using $\omega_n^{s'} \geq 0$,

$$|(E - X_n^s)\Omega_n - D| = \left| \sum_{s'} \omega_n^{s'} (X_n^{s'} - X_n^s) \right| \leq \sum_{s'} \omega_n^{s'} |X_n^{s'} - X_n^s| \leq \sum_{s'} \omega_n^{s'} \cdot \varepsilon E = \varepsilon E \Omega_n.$$

Separately, from Step 3, $D = E\Omega_n - \sum_{s'} \omega_n^{s'} X_n^{s'}$; since $X_n^{s'} \leq \varepsilon E$ for every s' , $\sum_{s'} \omega_n^{s'} X_n^{s'} \leq \varepsilon E \sum_{s'} \omega_n^{s'} = \varepsilon E\Omega_n$, so

$$D \geq E\Omega_n - \varepsilon E\Omega_n = E\Omega_n(1 - \varepsilon) > 0$$

(strictly positive since $\varepsilon < 1$).

Step 6: combine. Substituting the two bounds from Step 5 into the Step 4 expression,

$$\left| x_n^s - \frac{\omega_n^s}{\Omega_n} \right| = \frac{\omega_n^s}{D\Omega_n} |(E - X_n^s)\Omega_n - D| \leq \frac{\omega_n^s}{D\Omega_n} \cdot \varepsilon E\Omega_n = \frac{\varepsilon E \omega_n^s}{D} \leq \frac{\varepsilon E \omega_n^s}{E\Omega_n(1 - \varepsilon)} = \frac{\varepsilon}{1 - \varepsilon} \cdot \frac{\omega_n^s}{\Omega_n},$$

where the last inequality uses $D \geq E\Omega_n(1 - \varepsilon)$ from Step 5 (a smaller positive denominator makes the fraction larger, so dividing by the smaller bound gives an upper bound on the true ratio). Since $\hat{x}_n^s = \omega_n^s/\Omega_n$ by definition, this is the claimed bound; letting $\varepsilon \rightarrow 0$ makes the right-hand side vanish, so $x_n^s \rightarrow \hat{x}_n^s$. $\square$

Proof of Theorem 1. *Step 1: introduce $\hat{x}_n^s$ by adding and subtracting.* For each s , write the identity

$$x_n^s = \hat{x}_n^s + (x_n^s - \hat{x}_n^s)$$

(true trivially). Multiplying by m_i^s and summing over s ,

$$\sum_s x_n^s m_i^s = \sum_s \hat{x}_n^s m_i^s + \sum_s (x_n^s - \hat{x}_n^s) m_i^s.$$

By Assumption 1 and Lemma 3, $m_i^s = s_i^s$ exactly, so $\sum_s \hat{x}_n^s m_i^s = \sum_s \hat{x}_n^s s_i^s = \text{SCI}_{ni}$ by Definition 2. Define $R_{ni} \equiv \sum_s (x_n^s - \hat{x}_n^s) m_i^s$, so that

$$\sum_s x_n^s m_i^s = \text{SCI}_{ni} + R_{ni}. \quad (20)$$

This is the claimed decomposition; it remains to bound R_{ni} .

Step 2: bound each term $x_n^s - \hat{x}_n^s$. Because Assumptions 4 and 5 also hold by hypothesis, Proposition 3's bound applies directly: $|x_n^s - \hat{x}_n^s| \leq \frac{\varepsilon}{1 - \varepsilon} \hat{x}_n^s$ for every s .

Step 3: bound R_{ni} . By the triangle inequality applied to the sum defining R_{ni} ,

$$|R_{ni}| = \left| \sum_s (x_n^s - \hat{x}_n^s) m_i^s \right| \leq \sum_s |x_n^s - \hat{x}_n^s| m_i^s,$$

using $m_i^s \geq 0$, so $|(x_n^s - \hat{x}_n^s) m_i^s| = |x_n^s - \hat{x}_n^s| m_i^s$. Applying Step 2's bound termwise,

$$|R_{ni}| \leq \sum_s \frac{\varepsilon}{1 - \varepsilon} \hat{x}_n^s m_i^s = \frac{\varepsilon}{1 - \varepsilon} \sum_s \hat{x}_n^s m_i^s.$$

By Step 1, $\sum_s \hat{x}_n^s m_i^s = \text{SCI}_{ni}$ exactly, so

$$|R_{ni}| \leq \frac{\varepsilon}{1 - \varepsilon} \text{SCI}_{ni},$$

which is the claimed bound on R_{ni} .

Step 4: the limit. As $\varepsilon \rightarrow 0$, the bound $\frac{\varepsilon}{1-\varepsilon} \text{SCI}_{ni} \rightarrow 0$, so $R_{ni} \rightarrow 0$; taking this limit in (20) gives $\sum_s x_n^s m_i^s \rightarrow \text{SCI}_{ni}$, i.e. $\langle x_n, m_i \rangle \rightarrow \text{SCI}_{ni}$. $\square$

Proof of Lemma 4. *Step 1: the case $a \leq b$.* Suppose $a \leq b$, with $a, b \in [0, 1]$. Then by definition $\min(a, b) = a$. Compute the difference:

$$\min(a, b) - ab = a - ab = a(1 - b).$$

Since $a \geq 0$ (given) and $1 - b \geq 0$ (given $b \leq 1$), the product of two nonnegative numbers is nonnegative: $a(1 - b) \geq 0$, i.e. $\min(a, b) \geq ab$. Moreover $a(1 - b) = 0$ if and only if $a = 0$ or $1 - b = 0$ (a product of real numbers is zero iff at least one factor is zero), i.e. iff $a = 0$ or $b = 1$.

Step 2: the case $b \leq a$. By an identical argument with the roles of a and b exchanged, $\min(a, b) = b$, so $\min(a, b) - ab = b - ab = b(1 - a) \geq 0$ (since $b \geq 0, 1 - a \geq 0$), with equality iff $b = 0$ or $a = 1$.

Step 3: combine into a single inequality. Every pair $(a, b) \in [0, 1]^2$ falls into (at least) one of the two cases above, and in either case $\min(a, b) - ab \geq 0$. Hence $\min(a, b) \geq ab$ for all $a, b \in [0, 1]$, which is the first claim.

Step 4: the equality condition, sufficiency. Suppose $a \in \{0, 1\}$. If $a = 0$: since $a = 0 \leq b$ always holds (as $b \geq 0$), Case 1 applies, and its equality condition “ $a = 0$ or $b = 1$ ” is met by $a = 0$, so $\min(a, b) - ab = 0$. If $a = 1$: since $b \leq 1 = a$ always holds, Case 2 applies, and its equality condition “ $b = 0$ or $a = 1$ ” is met by $a = 1$, so $\min(a, b) - ab = 0$ again. By the symmetric argument, $b \in \{0, 1\}$ also forces $\min(a, b) - ab = 0$. So $a \in \{0, 1\}$ or $b \in \{0, 1\}$ is sufficient for equality.

Step 5: the equality condition, necessity. Conversely, suppose $a \notin \{0, 1\}$ and $b \notin \{0, 1\}$, i.e. $a, b \in (0, 1)$ strictly. If $a \leq b$ applies (Case 1), then $\min(a, b) - ab = a(1 - b)$ with $a > 0$ and $1 - b > 0$ (since $b < 1$), so the product is strictly positive: $\min(a, b) - ab > 0$, a strict inequality, not equality. If instead $b \leq a$ applies (Case 2), symmetrically $\min(a, b) - ab = b(1 - a) > 0$ since $b > 0, 1 - a > 0$. Either way, $a, b \in (0, 1)$ forces strict inequality, so equality requires $a \in \{0, 1\}$ or $b \in \{0, 1\}$. Combined with Step 4, equality holds if and only if $a \in \{0, 1\}$ or $b \in \{0, 1\}$.

Step 6: sum over sectors. Apply Steps 1–3 with $a = a_s = x_n^s \in [0, 1]$ and $b = b_s = m_i^s \in [0, 1]$ for each $s = 1, \dots, S$ (each is a share, hence in $[0, 1]$), giving $\min(x_n^s, m_i^s) \geq x_n^s m_i^s$ termwise. Summing this inequality over $s = 1, \dots, S$ preserves the direction (a sum of nonnegative gaps is nonnegative):

$$\sum_s \min(x_n^s, m_i^s) \geq \sum_s x_n^s m_i^s.$$

By Lemma 2, the left-hand side equals TCI_{ni} , and the right-hand side is $\langle x_n, m_i \rangle$ by definition, so $\text{TCI}_{ni} \geq \langle x_n, m_i \rangle$. Equality in the sum holds if and only if every one of the S termwise inequalities is an equality (since each term’s gap is individually nonnegative, the sum of gaps is zero iff every gap is zero), i.e. if and only if, for every s , $x_n^s \in \{0, 1\}$ or $m_i^s \in \{0, 1\}$ (Steps 4–5). $\square$

Proof of Corollary 1. *Step 1: isolate SCI_{ni} .* Rearranging Theorem 1’s decomposition $\langle x_n, m_i \rangle =$

$\text{SCI}_{ni} + R_{ni}$ by subtracting R_{ni} from both sides,

$$\text{SCI}_{ni} = \langle x_n, m_i \rangle - R_{ni}.$$

Step 2: apply the dominance lemma. By Lemma 4, $\langle x_n, m_i \rangle \leq \text{TCI}_{ni}$. Combining with Step 1,

$$\text{SCI}_{ni} = \langle x_n, m_i \rangle - R_{ni} \leq \text{TCI}_{ni} - R_{ni} \leq \text{TCI}_{ni} + |R_{ni}|,$$

using $-R_{ni} \leq |R_{ni}|$. This is the first claim.

Step 3: the small-exporter limit. Theorem 1 gives $|R_{ni}| \leq \frac{\varepsilon}{1-\varepsilon} \text{SCI}_{ni} \rightarrow 0$ as $\varepsilon \rightarrow 0$ (since SCI_{ni} is a fixed number in $[0, 1]$, not itself a function of ε , once n, i are fixed and ε indexes only how good the small-exporter approximation is). Taking $\varepsilon \rightarrow 0$ in Step 2 gives, in the limit, $\text{SCI}_{ni} \leq \text{TCI}_{ni}$ exactly.

Step 4: the equality condition. In the limit, equality $\text{SCI}_{ni} = \text{TCI}_{ni}$ holds if and only if $\langle x_n, m_i \rangle = \text{TCI}_{ni}$ exactly (Step 1 becomes an equality with $R_{ni} = 0$, so the Step 2 inequality binds if and only if the dominance-lemma inequality it was built from binds). By Lemma 4, this holds if and only if, for every sector s , $x_n^s \in \{0, 1\}$ or $m_i^s \in \{0, 1\}$, i.e. sector by sector, the exporter is either entirely absent from or entirely concentrated in that sector, or the importer is. $\square$

Proof sketch of Corollary 2. This result combines a cited limit theorem for the Fréchet distribution with the propositions proved above; we are explicit about which parts are proved here and which are cited.

Step 1 (cited, not re-derived): complete specialization as $\theta \rightarrow \infty$. Fix a sector s . Generically, the N numbers $\{T_m^s w_m^{-\theta}\}_{m=1}^N$ have a unique largest value as $\theta \rightarrow \infty$ (ties across countries are a measure-zero coincidence in the parameters $\{T_m^s, w_m\}$). It is a standard property of the Fréchet-based EK trade-share formula (Eaton and Kortum, 2002) that, as the dispersion parameter θ that indexes comparative advantage grows without bound, competition becomes winner-take-all: the market share of whichever country has the (generically unique) lowest effective unit cost in sector s converges to 1, and every other country's share converges to 0. In our notation, this means $\omega_n^s \rightarrow 1$ for the cost-minimizing country and $\omega_n^s \rightarrow 0$ for every other n , for each sector s separately. We cite this extreme-value limit rather than re-derive it, as it is a property of the Fréchet distribution underlying (1) and not specific to this paper's constructions.

Step 2 (proved here): the export share inherits the same limit. Maintain Assumption 3 as $\theta \rightarrow \infty$, so Proposition 3's exact formula $x_n^s = \omega_n^s (E^s - X_n^s) / \sum_{s'} \omega_n^{s'} (E^{s'} - X_n^{s'})$ continues to hold at every θ . Suppose country n is the cost-minimizer in exactly one sector, say $s = s^*(n)$ (the generic case, since $\{T_m^s w_m^{-\theta}\}_m$ has a unique maximizer sector-by-sector and these need not coincide across n and s). By Step 1, $\omega_n^{s^*(n)} \rightarrow 1$ and $\omega_n^s \rightarrow 0$ for every $s \neq s^*(n)$. Substituting into the Proposition 3 formula: every term in the denominator's sum with $s' \neq s^*(n)$ has $\omega_n^{s'} \rightarrow 0$ and hence vanishes (since $X_n^{s'}$ remains bounded as $\theta \rightarrow \infty$, being a demand-side primitive unaffected by the supply-side limit), while the $s' = s^*(n)$ term converges to $1 \cdot (E^{s^*(n)} - X_n^{s^*(n)})$. Hence the denominator converges to $E^{s^*(n)} - X_n^{s^*(n)}$, a strictly positive constant, and:

- for $s = s^*(n)$: numerator $\omega_n^s(E^s - X_n^s) \rightarrow 1 \cdot (E^{s^*(n)} - X_n^{s^*(n)})$, matching the denominator, so $x_n^{s^*(n)} \rightarrow 1$;
- for $s \neq s^*(n)$: numerator $\omega_n^s(E^s - X_n^s) \rightarrow 0 \cdot (E^s - X_n^s) = 0$, so $x_n^s \rightarrow 0$.

Hence $x_n^s \rightarrow \{0, 1\}$ for every s , degenerating to a point mass at $s^*(n)$.

Step 3 (proved here): the import share inherits the same limit. By Lemma 3 (which holds under the maintained Assumption 1), $m_i^s = s_i^s$ exactly at every θ ; this identity does not involve θ at all (it only used Assumption 1 on the domestic trade share), so it continues to hold in the limit. Since $s_i^s = X_i^s/X_i$ is a demand-side primitive independent of θ , the limiting behavior of m_i^s as $\theta \rightarrow \infty$ is simply whatever s_i^s already is; no degeneracy is forced on the import side by $\theta \rightarrow \infty$ *unless* the underlying demand primitives themselves are degenerate. For the statement of this corollary we consider the case where they are (the fully Ricardian limit on both sides of the market, i.e. complete specialization in *consumption* as well as production, a standard companion idealization); under that additional idealization $m_i^s \rightarrow \{0, 1\}$ by the same point-mass argument as Step 2.

Step 4: assemble. By Steps 2–3, $x_n^s, m_i^s \rightarrow \{0, 1\}$ for every s . By Lemma 4, the equality condition “ $x_n^s \in \{0, 1\}$ or $m_i^s \in \{0, 1\}$ ” then holds, in the limit, for every s , so $\langle x_n, m_i \rangle \rightarrow \text{TCI}_{ni}$ (Lemma 4 applied with equality termwise). Combining with Theorem 1 in the $\varepsilon \rightarrow 0$ limit, $\text{SCI}_{ni} \rightarrow \langle x_n, m_i \rangle \rightarrow \text{TCI}_{ni}$, which is the claim. $\square$

Proof of Proposition 4. *Step 1: importer access.* By the definition of ω_m^s in (3), $T_m^s w_m^{-\theta} = \omega_m^s \Phi^s$ for every m . Under Assumption 6,

$$\Phi_i^s = \sum_m T_m^s w_m^{-\theta} d_{mi}^{-\theta} = \Phi^s \sum_m \omega_m^s d_{mi}^{-\theta} = \Phi^s a_i^s,$$

using (7). Since $d_{ii} = 1$ and $\omega_i^s > 0$, $a_i^s > 0$.

Step 2: sector-level trade shares. Substituting Step 1 into (1) and applying Assumption 7,

$$\pi_{ni}^s = \frac{T_n^s w_n^{-\theta} d_{ni}^{-\theta}}{\Phi_i^s} = \frac{\omega_n^s \Phi^s d_{ni}^{-\theta}}{\Phi^s a_i} = \frac{d_{ni}^{-\theta}}{a_i} \omega_n^s.$$

Step 3: aggregation. By (2), $X_{ni} = X_i \sum_s \pi_{ni}^s s_i^s$. Substituting Step 2 and $\omega_n^s = \Omega_n \hat{x}_n^s$ from (4),

$$X_{ni} = X_i \frac{d_{ni}^{-\theta}}{a_i} \Omega_n \sum_s \hat{x}_n^s s_i^s = \frac{X_i}{a_i} \Omega_n d_{ni}^{-\theta} \text{SCI}_{ni},$$

the last equality by Definition 2. $\square$

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