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Commodity Price Shocks, Fiscal Capacity, and Monetary Policy in a Currency Union

Mathematical Companion

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This companion states, step by step, the mathematics behind the paper: the model's optimality conditions and their derivation, the steady state and its calibration, the first- and second-order solutions, the welfare criterion, the experiments, and the estimation of the commodity price factor and of member exposures. Economic interpretation is in the paper and is not repeated here. Equation numbers refer to this document unless marked as the paper's.

1 Notation and Conventions

Time is quarterly. For a strictly positive variable X_t with steady state $\bar{X}$, $\hat{x}_t \equiv \log X_t - \log \bar{X}$; for a variable that can be zero or negative (asset positions, transfers, revenue, the import tax, the price factors), $\hat{x}_t \equiv X_t - \bar{X}$. The headline price index P_t is the numeraire, and relative prices are

$$q_t = \frac{P_t^N}{P_t}, \quad s_t^X = \frac{P_t^X}{P_t}, \quad s_t^{F*} = \frac{P_t^{F*}}{P_t}, \quad s_t^F = \frac{P_t^F}{P_t}, \quad w_t = \frac{W_t}{P_t}, \quad p_t^h = \frac{P_t^h}{P_t}, \quad (1)$$

with $\Pi_t = P_t/P_{t-1}$ and $\Pi_t^N = P_t^N/P_{t-1}^N$, so that

$$q_t = q_{t-1} \Pi_t^N / \Pi_t. \quad (2)$$

Asset positions b_t^p, b_t^g , transfers Z_t , revenue T_t , the lump-sum tax T^{ls} and the non-commodity flow x^o are in units of steady-state output $\bar{Y}$. Household types are $h \in \{R, H\}$ with masses $1 - \mu_H$ and μ_H .

2 Households

2.1 Consumption Composite and Price Indices

The composite of type h is

$$C_t^h = \left[(1 - \gamma_h)^{\frac{1}{\eta}} (C_t^{h,N})^{\frac{\eta-1}{\eta}} + \gamma_h^{\frac{1}{\eta}} (C_t^{h,F})^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}. \quad (3)$$

Minimizing $P_t^N C_t^{h,N} + P_t^F C_t^{h,F}$ subject to a given C_t^h gives the demands and the cost-of-living index

$$C_t^{h,N} = (1 - \gamma_h) \left(\frac{P_t^N}{P_t^h} \right)^{-\eta} C_t^h = (1 - \gamma_h) \left(\frac{q_t}{p_t^h} \right)^{-\eta} C_t^h, \quad (4)$$

$$C_t^{h,F} = \gamma_h \left(\frac{P_t^F}{P_t^h} \right)^{-\eta} C_t^h = \gamma_h \left(\frac{s_t^F}{p_t^h} \right)^{-\eta} C_t^h, \quad (5)$$

$$(p_t^h)^{1-\eta} = (1 - \gamma_h) q_t^{1-\eta} + \gamma_h (s_t^F)^{1-\eta}. \quad (6)$$

Substituting (4)–(5) into expenditure yields the identity

$$p_t^h C_t^h = q_t C_t^{h,N} + s_t^F C_t^{h,F}. \quad (7)$$

The headline index has weight $\bar{\gamma} = \mu_H \gamma_H + (1 - \mu_H) \gamma_R$, and because it is the numeraire,

$$1 = (1 - \bar{\gamma}) q_t^{1-\eta} + \bar{\gamma} (s_t^F)^{1-\eta}. \quad (8)$$

Aggregates over types are $C_t^N = \mu_H C_t^{H,N} + (1 - \mu_H) C_t^{R,N}$, $C_t^F = \mu_H C_t^{H,F} + (1 - \mu_H) C_t^{R,F}$, $L_t = \mu_H L_t^H + (1 - \mu_H) L_t^R$, and total expenditure relative to P_t is $C_t \equiv \mu_H p_t^H C_t^H + (1 - \mu_H) p_t^R C_t^R = q_t C_t^N + s_t^F C_t^F$ by (7).

2.2 Labor Supply

Period utility is $\log C_t^h - (L_t^h)^{1+\varphi}/(1+\varphi)$. With nominal budget multiplier $\lambda_t^h = 1/(P_t^h C_t^h)$ for either type, the first-order condition for hours, $W_t \lambda_t^h = (L_t^h)^\varphi$, gives

$$\frac{w_t}{p_t^h} = C_t^h (L_t^h)^\varphi, \quad h \in \{R, H\}. \quad (9)$$

2.3 Ricardian Households

Ricardian households choose consumption, hours and bond holdings. The bond pays $R_t \Phi_t$, where the premium is taken as given:

$$\log \Phi_t = -\psi_b(b_t^p - \bar{b}) - \psi_x \log(s_t^X / \bar{s}^X). \quad (10)$$

The first-order condition for the bond, $\lambda_t^R = \beta R_t \Phi_t \mathbb{E}_t \lambda_{t+1}^R$, with $\lambda_t^R = 1/(P_t^R C_t^R)$ and $P_{t+1}^R / P_t^R = \Pi_{t+1} p_{t+1}^R / p_t^R$, is

$$1 = \beta R_t \Phi_t \mathbb{E}_t \left[\frac{C_t^R}{C_{t+1}^R} \frac{p_t^R}{\Pi_{t+1} p_{t+1}^R} \right]. \quad (11)$$

The aggregate private budget (both types, with firm profits, specific-factor income and transfers accruing to households, and the lump-sum tax paid by Ricardians), relative to P_t and in units of $\bar{Y}$, is

$$\frac{b_t^p}{R_t \Phi_t} = \frac{b_{t-1}^p}{\Pi_t} + \frac{q_t Y_t^N + (1 - \tau^X) s_t^X Y_t^X - s_t^F M_t - C_t}{\bar{Y}} + x^o + Z_t - T^{ls}. \quad (12)$$

2.4 Hand-to-Mouth Households

A share $\delta \in [\mu_H, 1]$ of transfers reaches hand-to-mouth households:

$$p_t^H C_t^H = w_t L_t^H + \delta Z_t \bar{Y} / \mu_H. \quad (13)$$

With $Z \equiv 0$, (9) and (13) give $w_t / p_t^H = C_t^H (L_t^H)^\varphi$ and $C_t^H = w_t L_t^H / p_t^H$, hence $(L_t^H)^{1+\varphi} = 1$: $L_t^H = 1$ and $C_t^H = w_t / p_t^H$ in every period.

3 Firms

3.1 Non-Traded Sector: Cost Minimization

Firm j produces $Y_t^N(j) = A_t^N L_t^N(j)^{1-\chi} M_t(j)^\chi$. Minimizing $W_t L + P_t^F M$ gives the input ratio and nominal marginal cost

$$M_t(j) = \frac{\chi}{1-\chi} \frac{W_t}{P_t^F} L_t^N(j) = \frac{\chi}{1-\chi} \frac{w_t}{s_t^F} L_t^N(j), \quad (14)$$

$$MC_t = \frac{1}{A_t^N} \left(\frac{W_t}{1-\chi} \right)^{1-\chi} \left(\frac{P_t^F}{\chi} \right)^\chi, \quad mc_t \equiv \frac{MC_t}{P_t^N} = \frac{1}{q_t A_t^N} \left(\frac{w_t}{1-\chi} \right)^{1-\chi} \left(\frac{s_t^F}{\chi} \right)^\chi. \quad (15)$$

The conditional labor demand $L_t^N(j) = (1-\chi) MC_t Y_t^N(j) / W_t$ is implied by (14), (15) and the technology. Since the input ratio is common to all firms, $L_t^N = \int L_t^N(j) dj$ and $M_t = \int M_t(j) dj$ satisfy the same ratio.

3.2 Non-Traded Sector: Price Setting

Demand for variety j is $Y_t^N(j) = (P_t^N(j)/P_t^N)^{-\epsilon} Y_t^N$. A firm resets its price with probability $1 - \xi_p$. Ricardian households own firms, so future nominal profits are discounted by $\beta^k \lambda_{t+k}^R / \lambda_t^R$. The reset price $\tilde{P}_t$ solves

$$\max_{\tilde{P}_t} \mathbb{E}_t \sum_{k \geq 0} (\beta \xi_p)^k \lambda_{t+k}^R \left(\tilde{P}_t - MC_{t+k} \right) \left(\frac{\tilde{P}_t}{P_{t+k}^N} \right)^{-\epsilon} Y_{t+k}^N. \quad (16)$$

The first-order condition,

$$\mathbb{E}_t \sum_{k \geq 0} (\beta \xi_p)^k \lambda_{t+k}^R (P_{t+k}^N)^\epsilon Y_{t+k}^N \left[(\epsilon - 1) \tilde{P}_t - \epsilon MC_{t+k} \right] = 0, \quad (17)$$

is written recursively with $\tilde{p}_t = \tilde{P}_t / P_t^N$. Dividing by $(P_t^N)^\epsilon$ and using $\lambda_t^R P_t^N = q_t / (p_t^R C_t^R)$,

$$x_t^1 = \frac{q_t}{p_t^R C_t^R} Y_t^N mc_t + \beta \xi_p \mathbb{E}_t [(\Pi_{t+1}^N)^\epsilon x_{t+1}^1], \quad (18)$$

$$x_t^2 = \frac{q_t}{p_t^R C_t^R} Y_t^N + \beta \xi_p \mathbb{E}_t [(\Pi_{t+1}^N)^{\epsilon-1} x_{t+1}^2], \quad (19)$$

$$\tilde{p}_t = \frac{\epsilon}{\epsilon - 1} \frac{x_t^1}{x_t^2}. \quad (20)$$

The price index $(P_t^N)^{1-\epsilon} = \xi_p (P_{t-1}^N)^{1-\epsilon} + (1 - \xi_p) \tilde{P}_t^{1-\epsilon}$ becomes

$$1 = \xi_p (\Pi_t^N)^{\epsilon-1} + (1 - \xi_p) \tilde{p}_t^{1-\epsilon}. \quad (21)$$

Price dispersion $\Delta_t = \int (P_t^N(j)/P_t^N)^{-\epsilon} dj$ evolves as

$$\Delta_t = \xi_p (\Pi_t^N)^\epsilon \Delta_{t-1} + (1 - \xi_p) \tilde{p}_t^{-\epsilon}, \quad (22)$$

and integrating firm output over j with the common input ratio gives

$$Y_t^N \Delta_t = A_t^N (L_t^N)^{1-\alpha_X} M_t^X. \quad (23)$$

3.3 Export Sector

With $Y_t^X = A_t^X (L_t^X)^{\alpha_X}$, the specific factor normalized into A_t^X , and a royalty τ^X on revenue, profit maximization $\max_L (1 - \tau^X) P_t^X A_t^X L^{\alpha_X} - W_t L$ gives

$$w_t = (1 - \tau^X) s_t^X \alpha_X A_t^X (L_t^X)^{\alpha_X-1}. \quad (24)$$

The residual $(1 - \alpha_X)(1 - \tau^X) s_t^X Y_t^X$ is specific-factor income, included in (12).

4 World Prices, Exposure, Government and Policy

4.1 Commodity Prices

With $p_t^k = \Lambda_k f_t + u_t^k$ the log world price of commodity k in union currency,

$$f_t = \rho_f f_{t-1} + \nu_t^f, \quad u_t^k = \rho_k u_{t-1}^k + \nu_t^k, \quad (25)$$

$$s_t^X = \frac{\exp(\sum_k \theta_k^X p_t^k)}{P_t}, \quad s_t^{F*} = \frac{\exp(\sum_k \theta_k^F p_t^k)}{P_t}, \quad s_t^F = (1 + \tau_t^M) s_t^{F*}. \quad (26)$$

Productivity follows $\log A_t^N = \rho_{aN} \log A_{t-1}^N + \nu_t^N$ and $\log(A_t^X/\bar{A}^X) = \rho_{aX} \log(A_{t-1}^X/\bar{A}^X) + \nu_t^X$. All innovations are i.i.d., mutually independent, with variance σ_ν^2 : $\Sigma_\nu = \sigma_\nu^2 I$.

4.2 Net Exposure

With $X/Y = \bar{s}^X \bar{Y}^X / \bar{Y}$ and $M/Y = \bar{s}^{F*}(\bar{C}^F + \bar{M}) / \bar{Y}$,

$$n(i) = \frac{X}{Y} \theta^X \cdot \Lambda - \frac{M}{Y} \theta^F \cdot \Lambda. \quad (27)$$

In the stylized union ($x^o = 0$, $b^p = \bar{b}$, $b^g = \bar{b}^g = 0$), the steady-state balance of payments (36) gives $X/Y = M/Y \equiv s$, so $n = s(\theta^X - \theta^F) \cdot \Lambda$. With $K = 2$ and $\Lambda = (1, 0)$, $\theta \cdot \Lambda = \theta_1$.

4.3 Government

$$T_t = [\tau^X s_t^X Y_t^X + \tau_t^M s_t^{F*} (C_t^F + M_t)] / \bar{Y}, \quad (28)$$

$$\frac{b_t^g}{R_t \Phi_t^g} = \frac{b_{t-1}^g}{\Pi_t} + T_t + T^{ls} - q_t g_y - Z_t, \quad \log \Phi_t^g = -\psi_b^g(b_t^g - \bar{b}^g) - \psi_x^g \log(s_t^X / \bar{s}^X), \quad (29)$$

$$Z_t = \zeta_x \log \frac{s_t^X}{\bar{s}^X} + \zeta_f \log \frac{s_t^{F*}}{\bar{s}^{F*}} + \zeta_d \log \frac{s_t^{F*}}{s_{t-1}^{F*}} + \zeta_b(b_t^g - \bar{b}^g), \quad (30)$$

$$\tau_t^M = \bar{\tau}^M + \sigma_f \log \frac{s_t^{F*}}{\bar{s}^{F*}}. \quad (31)$$

Government purchases are $G = g_y \bar{Y}$ units of the non-traded good.

Stability of government assets. Near the steady state ($R\Phi^g = 1/\beta$, $\Pi = 1$), holding all other terms of (29) at steady state and substituting (30),

$$\beta \hat{b}_t^g = \hat{b}_{t-1}^g - \zeta_b \hat{b}_t^g \implies \hat{b}_t^g = \frac{\hat{b}_{t-1}^g}{\beta + \zeta_b}, \quad (32)$$

which is stable iff $\beta + \zeta_b > 1$, i.e. $\zeta_b > 1 - \beta$.

4.4 Monetary Policy

For a member solved alone,

$$\log R_t = \log(1/\beta) + \kappa [\phi_\pi \log \Pi_t + \phi_y \log(Y_t/\bar{Y})], \quad Y_t = q_t Y_t^N + s_t^X Y_t^X - s_t^{F*} M_t. \quad (33)$$

In a union of members $i = 1, \dots, M$ with weights ω_i summing to one, each member's rule is replaced by $\log R_t(i) = r_t^U$, and

$$r_t^U = \log(1/\beta) + \kappa \sum_i \omega_i [\phi_\pi \log \Pi_t(i) + \phi_y \log(Y_t(i)/\bar{Y}(i))]. \quad (34)$$

4.5 Market Clearing and the Balance of Payments

$$L_t = L_t^N + L_t^X, \quad Y_t^N = C_t^N + g_y \bar{Y}. \quad (35)$$

Derivation of the balance of payments. Add (12) and (29). Transfers Z_t and T^{ls} cancel. Using $C_t = q_t C_t^N + s_t^F C_t^F$ and (35), $q_t Y_t^N - C_t - q_t g_y \bar{Y} = -s_t^F C_t^F$. The royalty terms combine as $(1 - \tau^X) s_t^X Y_t^X + \tau^X s_t^X Y_t^X = s_t^X Y_t^X$, and with $s_t^F = (1 + \tau_t^M) s_t^{F*}$, $-s_t^F (C_t^F + M_t) + \tau_t^M s_t^{F*} (C_t^F + M_t) = -s_t^{F*} (C_t^F + M_t)$. Hence

$$\frac{b_t^p}{R_t \Phi_t} + \frac{b_t^g}{R_t \Phi_t^g} = \frac{b_{t-1}^p + b_{t-1}^g}{\Pi_t} + \frac{s_t^X Y_t^X - s_t^{F*} (C_t^F + M_t)}{\bar{Y}} + x^o. \quad (36)$$

This equation is implied by the others (Walras' law) and is not included in the system; it is checked as a residual.

Equation count. With $K = 2$ commodities the member system has 46 equations in 46 variables; the complete list is Appendix A of the paper.

5 Steady State and Calibration

5.1 Closed-Form Restrictions

With zero innovations and constant variables: $\Pi = \Pi^N = 1$. Then (21) gives $\tilde{p} = 1$, (22) gives $\Delta = 1$, and (18)–(20) give $x^1/x^2 = \bar{m}\bar{c}$ and $\tilde{p} = \frac{\epsilon}{\epsilon-1} \bar{m}\bar{c} = 1$:

$$\bar{m}\bar{c} = \frac{\epsilon - 1}{\epsilon}. \quad (37)$$

At $b^p = \bar{b}$, $s^X = \bar{s}^X$ the premium is $\Phi = 1$, and (11) gives $\bar{R} = 1/\beta$. With $Z = 0$, $\bar{L}^H = 1$. The steady-state balance of payments (36) with $\bar{b}^g = 0$, $\bar{R}\Phi = 1/\beta$ and $\Pi = 1$ reads $\beta\bar{b} = \bar{b} + (X/Y - M/Y) + x^o$, so

$$\frac{X}{Y} - \frac{M}{Y} = -(1 - \beta)\bar{b} - x^o. \quad (38)$$

5.2 Numerical Solution

The remaining steady state solves the static system with four auxiliary unknowns, $\bar{Y}$, T^{ls} , $\bar{s}^X$ and $\bar{s}^{F*}$, closed by

$$\bar{Y} = Y, \quad \bar{s}^X = s^X, \quad \bar{s}^{F*} = s^{F*}, \quad b^g = \bar{b}^g, \quad (39)$$

the last pinning T^{ls} . Positive variables are solved in logs with a bounded Newton-type method, from a continuation path in the parameters when a direct solve fails.

5.3 Calibration of the Budget Shares

Given target export share ω_x , the budget shares are $(\gamma_R, \gamma_H) = c(\gamma_R^0, \gamma_H^0)$ with the ratio γ_H/γ_R fixed, and c solves the scalar equation

$$\frac{\bar{s}^X(c) \bar{Y}^X(c)}{\bar{Y}(c)} = \omega_x \quad (40)$$

by root-finding, each evaluation being a full steady-state solve. In the country calibration $x^o = -(X/Y - M/Y)$ with $\bar{b} = \bar{b}^g = 0$, so by (36) matching X/Y also matches M/Y .

6 First-Order Solution

6.1 Canonical Form

Stack the $n = 46$ variables in y_t (log or level deviations as in Section 1) and the shocks in ε_t . Write the system as $F(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) = 0$. The first-order approximation is

$$A \mathbb{E}_t y_{t+1} + B y_t + C y_{t-1} + D \varepsilon_t = 0, \quad (41)$$

where $A = \partial F / \partial y_{t+1}$ etc. are symbolic derivatives evaluated at the steady state, with the chain rule $\partial F / \partial \hat{x} = \bar{X} \partial F / \partial X$ for log variables. Only (11), (18) and (19) contain leads, so rank $A = 3$.

6.2 Solution by QZ

Let f index the forward-looking variables (the six variables appearing with a lead: $C^R, \Pi, p^R, \Pi^N, x^1, x^2$). With $z_t = [y_t; \mathbb{E}_t y_{t+1}^f]$ and expectation errors $\eta_t = y_t^f - \mathbb{E}_{t-1} y_t^f$, the system becomes

$$\Gamma_0 z_t = \Gamma_1 z_{t-1} + \Psi \varepsilon_t + \Pi \eta_t, \quad (42)$$

solved by the QZ decomposition of (Γ_0, Γ_1) [Sims(2002)]. A unique stable solution exists iff the number of generalized eigenvalues outside the unit circle equals the number of independent expectation errors (six). The solution is

$$y_t = P y_{t-1} + Q \varepsilon_t, \quad (43)$$

verified by $AP^2 + BP + C = 0$ and $(AP + B)Q + D = 0$, and against a stacked-time solve of $Ay_{t+1} + By_t + Cy_{t-1} + D\varepsilon_t = 0$ for $t = 0, \dots, T-1$ with $y_{-1} = y_T = 0$.

6.3 Union System

For members $i = 1, \dots, M$ with systems (A_i, B_i, C_i, D_i) , the stacked state is $y_t^U = [y_t(1); \dots; y_t(M); r_t^U]$ and

$$\mathcal{A} = \text{blkdiag}(A_i, 0), \quad \mathcal{B} = \text{blkdiag}(B_i, 1), \quad \mathcal{C} = \text{blkdiag}(C_i, 0), \quad (44)$$

after two modifications: the row of each member's own rule (33) is replaced by $\hat{R}_t(i) - r_t^U = 0$, and the last row is (34) in deviations, $r_t^U - \kappa \sum_i \omega_i (\phi_\pi \hat{\pi}_t(i) + \phi_y \hat{y}_t(i)) = 0$. Each member keeps its own shocks.

6.4 Impulse Responses and the Imposed Component

For an innovation $\nu_0^f = 0.01$, $y_t = P^t Q e_f 0.01$. For member i , let $\hat{Y}_t^U(i)$ be its output response in the union and $\hat{Y}_t^O(i)$ its response when it is the only member (same κ , same rule, its own aggregates). With the gap $g_t(i) = \hat{Y}_t^U(i) - \hat{Y}_t^O(i)$, $t = 0, \dots, 23$,

$$\text{imposed component} = 10^4 \max_t |g_t(i)| \quad (\text{basis points}), \quad (45)$$

$$\text{share} = \max_t |g_t(i)| / \max_t |\hat{Y}_t^O(i)|, \quad (46)$$

$$\text{duration} = 1 + \max\{t : |g_t(i)| \geq 0.1 \max_s |g_s(i)|\}. \quad (47)$$

7 Second-Order Solution and Welfare

7.1 Policy Function

With $x = y_{t-1}$ and $e = \varepsilon_t$, the second-order policy function [Schmitt-Grohé and Uribe(2004)] is

$$y_t = Px + Qe + \frac{1}{2}[\Delta + G_{xx}(x \otimes x) + 2G_{xe}(x \otimes e) + G_{ee}(e \otimes e)]. \quad (48)$$

Let H be the Hessian of F with respect to $v = (y_{t+1}, y_t, y_{t-1}, \varepsilon_t)$, $M_x = [P^2; P; I; 0]$, $M_e = [PQ; Q; 0; I]$ and $M_{e'} = [Q; 0; 0; 0]$. Matching second-order terms gives

$$(AP + B)G_{xx} + AG_{xx}(P \otimes P) = -H[M_x, M_x], \quad (49)$$

$$(AP + B)G_{xe} = -AG_{xx}(P \otimes Q) - H[M_x, M_e], \quad (50)$$

$$(AP + B)G_{ee} = -AG_{xx}(Q \otimes Q) - H[M_e, M_e], \quad (51)$$

$$(AP + A + B)\Delta = -AG_{ee} \text{vec}(\Sigma_\nu) - H[M_{e'}, M_{e'}] : \Sigma_\nu, \quad (52)$$

where $H[U, V]$ denotes the bilinear form of H applied to the columns of U and V . The first equation is a Sylvester equation. H is computed by central finite differences of the symbolic first derivatives. In the union, H is block diagonal across members, and the union rule is linear.

7.2 Moments

The first-order unconditional covariance solves the Lyapunov equation

$$\Sigma_y = P\Sigma_y P' + Q\Sigma_\nu Q'. \quad (53)$$

Taking expectations of the policy function ($\mathbb{E}[x \otimes e] = 0$) gives the second-order mean

$$\mu = P\mu + \frac{1}{2}[\Delta + G_{xx} \text{vec}(\Sigma_y) + G_{ee} \text{vec}(\Sigma_\nu)] \implies \mu = \frac{1}{2}(I - P)^{-1}[\Delta + G_{xx} \text{vec}(\Sigma_y) + G_{ee} \text{vec}(\Sigma_\nu)]. \quad (54)$$

Every element of μ is proportional to σ_ν^2 .

7.3 Welfare

For type h , with $c^h = \hat{c}^h$ and $l^h = \hat{l}^h$, $\mathbb{E} \log C^h = \log \bar{C}^h + \mathbb{E}[c^h]$ and, expanding $(L^h)^{1+\varphi} = (\bar{L}^h)^{1+\varphi} e^{(1+\varphi)l^h}$ to second order,

$$\mathcal{W}^h = \log \bar{C}^h + \mu_{c^h} - \frac{(\bar{L}^h)^{1+\varphi}}{1+\varphi} \left[1 + (1+\varphi)\mu_{l^h} + \frac{(1+\varphi)^2}{2}\Sigma_{y,l^h l^h} \right], \quad (55)$$

where $\mathbb{E}[(l^h)^2]$ is replaced by $\text{var}(l^h)$ because $(\mathbb{E}l^h)^2 = O(\sigma_\nu^4)$. Member welfare is $\mathcal{W} = \omega_H \mathcal{W}^H + (1 - \omega_H) \mathcal{W}^R$ with $\omega_H = \mu_H$ in the baseline. A welfare difference is reported as

$$\text{CE} = 10^4(e^{\Delta \mathcal{W}} - 1) \quad (\text{basis points}), \quad (56)$$

since a permanent proportional change x in consumption changes period utility by $\log(1+x)$.

7.4 Validity Ratio

$$v = \max_{z \in \{C^R, C^H, L^R, L^H\}} \frac{|\mu_z|}{\sqrt{\Sigma_{y,zz}}}. \quad (57)$$

Because $\mu_z \propto \sigma_\nu^2$ and $\sqrt{\Sigma_{y,zz}} \propto \sigma_\nu$, $v \propto \sigma_\nu$; welfare gains of a given policy are proportional to σ_ν^2 .

8 Experiments

8.1 Exposure Grid

A probe member with $\theta^X = (t, 1 - t)$, $\theta^F = (1 - t, t)$, $t \in \{0, \frac{1}{8}, \dots, 1\}$ and weight 0.02 is added to the stylized union; its trade share is calibrated as in Section 5.3, so $n = s(2t - 1)$.

8.2 Dispersion Decomposition

Two members with weights $\frac{1}{2}$ have $t_{\pm} = \frac{1}{2} \pm \frac{d}{2}$, so $n_{\pm} = \pm s d$ and the dispersion is the population standard deviation $s d$. With $\mathcal{W}_{\pm}^{\text{full}}(d)$ computed on the two-member union and $\mathcal{W}_{\pm}^{\text{spec}}(d)$ on a union of two identical members with $t_{\pm}$, the reported quantities are averages over the two members:

$$\bar{\mathcal{W}}^j(d) = \frac{1}{2} [\mathcal{W}_+^j(d) + \mathcal{W}_-^j(d)], \quad j \in \{\text{full}, \text{spec}\}, \quad (58)$$

$$\underbrace{\bar{\mathcal{W}}^{\text{full}}(d) - \bar{\mathcal{W}}^{\text{full}}(0)}_{\text{total}} = \underbrace{\bar{\mathcal{W}}^{\text{spec}}(d) - \bar{\mathcal{W}}^{\text{spec}}(0)}_{\text{specialization}} + \underbrace{\bar{\mathcal{W}}^{\text{full}}(d) - \bar{\mathcal{W}}^{\text{spec}}(d)}_{M(d)}, \quad (59)$$

using $\bar{\mathcal{W}}^{\text{full}}(0) = \bar{\mathcal{W}}^{\text{spec}}(0)$. Each term is converted with $100(e^{\cdot} - 1)$.

8.3 Persistence of a Level-Indexed Transfer

For $x_t = \rho x_{t-1} + \sigma e_t$:

$$\text{var}(x_t) = \frac{\sigma^2}{1 - \rho^2}, \quad \text{var}(x_t - x_{t-1}) = 2 \text{var}(x_t)(1 - \rho) = \frac{2\sigma^2}{1 + \rho}, \quad (60)$$

$$\frac{\text{var}(x_t)}{\text{var}(x_t - x_{t-1})} = \frac{1}{2(1 - \rho)}, \quad \frac{\text{sd}(x_t)}{\sigma} = \frac{1}{\sqrt{1 - \rho^2}}. \quad (61)$$

At $\rho = 0.977$ these equal 21.7 and 4.69. A crossover between grid points $\rho_0 < \rho_1$ with margins $m_0 > 0 > m_1$ is interpolated as $\rho^* = \rho_0 + (\rho_1 - \rho_0) m_0 / (m_0 - m_1)$.

8.4 Optimal Fiscal Rules

Positivity. $1 + \tau_t^M > 0$ for all realizations within k standard deviations of $\log s_t^{F*}$ requires

$$|\sigma_f| k \text{sd}(\log s_t^{F*}) < 1 + \bar{\tau}^M \iff |\sigma_f| < \frac{1 + \bar{\tau}^M}{k \text{sd}(\log s_t^{F*})}, \quad (62)$$

with $k = 4$ and sd from the first-order solution under no fiscal response.

Admissible region. Starting from $(\zeta_f, \sigma_f) = (0, 0)$, each coefficient is moved outward in steps of 0.5 (ζ_f) and 0.25 (σ_f), the other held at zero, until the model fails to solve, $v > 0.35$, or the positivity bound is reached. This defines the box $[\zeta_f^{lo}, \zeta_f^{hi}] \times [\sigma_f^{lo}, \sigma_f^{hi}]$.

Search. A 9×9 grid over the box; candidates with no solution or $v > 0.35$ are discarded; the optimum maximizes $\mathcal{W}$.

Single-instrument rules. Each of $\zeta_f \in \{-2, -1.75, \dots, 2\}$, $\zeta_d \in \{-1, -0.9, \dots, 1\}$ and $\sigma_f \in \{-1.5, -1.25, \dots, 1.5\}$ is evaluated with the other coefficients at zero, $\delta = 1$ and $v \leq 0.35$.

8.5 Welfare Weights

The comparison of Table 14 of the paper uses $\mathcal{W}(\omega_H) = \omega_H \mathcal{W}^H + (1 - \omega_H) \mathcal{W}^R$ for $\omega_H \in \{\mu_H, 0.70, 0.85, 1\}$.

9 Estimation

9.1 Pink Sheet Factor

Monthly nominal prices P_m^k are averaged to quarters, logged, and demeaned over 1995Q1–2024Q4; series with missing values are dropped, leaving $K = 60$. With the $T \times K$ matrix X and its singular value decomposition $X = USV'$,

$$\tilde{f} = U_{\cdot 1} S_1, \quad v_1 = V_{\cdot 1}, \quad \text{share} = \frac{S_1^2}{\sum_j S_j^2}, \quad (63)$$

with the sign chosen so that $\bar{v}_1 > 0$; $f = \tilde{f} / \text{sd}(\tilde{f})$ and $\Lambda_k = v_{1k} \text{sd}(\tilde{f})$. Residuals are $E = X - \tilde{f} v_1'$. For any series y , the AR(1) coefficient is estimated without a constant,

$$\hat{\rho} = \text{clip}\left(\frac{\sum_t y_{t-1} y_t}{\sum_t y_{t-1}^2}, 0, 0.995\right), \quad (64)$$

applied to f and to each column of E ; ρ_k is set to the median over k .

9.2 Commodity Terms-of-Trade Factor and Exposures

For each country the IMF database publishes, for exports and imports, a basket-weighted index (weights θ_k summing to one) and a GDP-weighted index (weights X_k/Y or M_k/Y). Both are geometric averages, so in logs

$$\log I_t^{X, \text{bk}} = \sum_k \theta_k^X p_t^k, \quad \log I_t^{X, \text{gdp}} = \sum_k \frac{X_k}{Y} p_t^k = \frac{X}{Y} \sum_k \theta_k^X p_t^k, \quad (65)$$

and analogously for imports. Monthly indices are averaged to quarters from 1980, logged and demeaned. The factor f is the first principal component of the panel of basket-weighted export indices across countries, standardized to unit variance; its persistence is $\text{corr}(f_t, f_{t-1})$. Each country's indices are projected on f :

$$b = \frac{f'x}{f'f} \quad \text{for } x \in \{\log I^{X, \text{gdp}}, \log I^{M, \text{gdp}}, \log I^{X, \text{bk}}, \log I^{M, \text{bk}}\}, \quad (66)$$

giving b^X, b^M, c^X, c^M . By the expressions above, $b^X = (X/Y) \theta^X \cdot \Lambda$ and $c^X = \theta^X \cdot \Lambda$ up to estimation error, so

$$n = b^X - b^M, \quad \frac{X}{Y} = \frac{b^X}{c^X} \text{ if } |c^X| > 0.05, \quad \frac{M}{Y} = \frac{b^M}{c^M} \text{ if } |c^M| > 0.05. \quad (67)$$

9.3 Moving Block Bootstrap

With T quarters and block length $\ell = 12$, each replication draws $\lceil T/\ell \rceil$ start dates uniformly from $\{0, \dots, T - \ell\}$, concatenates the blocks, truncates to T rows, and recomputes f and all loadings. With $B = 2000$ replications, the standard error of n is the standard deviation across replications and the 95% interval is the 2.5th–97.5th percentile range.

9.4 Two Weighting Sets

The same computation on the annual panel with historical rolling weights gives a second set of exposures; the rank correlation between the two sets is Spearman's.

10 Country Calibration

For each member, with $\Lambda = (1, 0)$:

$$\theta^X = (c^X, 1 - c^X), \quad \theta^F = (c^M, 1 - c^M) \quad (\text{clipped to } [0, 1]), \quad (68)$$

$$\frac{X}{Y}, \frac{M}{Y} \text{ clipped to } [0.005, 0.60]; \quad \frac{X}{Y} \equiv \frac{M}{Y} \text{ when } c^X \text{ is below the threshold,} \quad (69)$$

$$x^o = -\left(\frac{X}{Y} - \frac{M}{Y}\right), \quad \chi = \min\left(0.12, 0.70 \frac{M}{Y}\right), \quad \tau^X = \begin{cases} 0.45, & n > 0.05 \\ 0.15, & \text{otherwise} \end{cases}, \quad (70)$$

and the budget shares are calibrated as in Section 5.3 from $(\gamma_R^0, \gamma_H^0) = (0.30, 0.45)$ to $\omega_x = X/Y$. By (27), the model exposure is $(X/Y)c^X - (M/Y)c^M = b^X - b^M = n$ exactly when both shares are measured. Weights are $\omega_i = \text{GDP}_i / \sum_{j \in \text{bloc}} \text{GDP}_j$ with GDP in current dollars averaged over 2015–2024. The debt elasticity is set by

$$\psi_b = \psi_b^g = \min\{0.01 \cdot 2^m : \text{the member has a unique stable solution}\}, \quad 0.01 \cdot 2^m \leq 0.32. \quad (71)$$

Within each bloc, the correlation reported in the paper is the Pearson correlation between the imposed component and $|n(i) - \sum_j \omega_j n(j)|$, and the rank correlation with trade is Spearman's between the imposed component and $X/Y + M/Y$.

11 Sensitivity Runs

Labor elasticity of export output. All results of the paper's Table 13 are recomputed with $\alpha_X \in \{0.02, 0.05, 0.20, 0.70\}$, each in a separate run with every other parameter at baseline.

Debt elasticity. $\psi_b = \psi_b^g$ is set to a common value $\psi \in \{0.01, 0.02, \dots, 0.64\}$ (doubling) for every member of a bloc; the bloc is solved at each value, and the imposed component is recorded for four members fixed in advance.

Validity cap and positivity horizon. The optimal-rule search of Section 7.4 is repeated for v -caps $\{0.25, 0.35, 0.50\}$ and $k \in \{3, 4, 5\}$ on the same 9×9 grid.

Shock-size scaling. Welfare gains are recomputed at $\sigma_\nu \in \{0.003, 0.006, 0.010\}$ and compared after multiplying by $(0.006/\sigma_\nu)^2$.

References

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